OMEGALEPH INSTITUTE FOR ADVANCED EDUCATION

A Festschrift in Honor of Gustavo Haddad Braga, the First Gold Medal for Brazil, Now the First Among the Ibero-American Countries in the History of the IPhOs to Receive Gold Medal - IPhO 42nd in Bangkok Thailand, 2011

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International Physics Olympiads 1967-2011

Part 4 - XLI-XLII - IPhO 2010-2011

OMEGALEPH

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International Physics Olympiads 2010-2011

IPhO 2010-2011

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A Festschrift in Honor of Gustavo Haddad Braga, the First Gold Medal for Brazil, Now the First Among the Ibero-American Countries in the History of the IPhOs to Receive Gold Medal - IPhO 42nd in Bangkok Thailand, 2011

The 41st International Physics Olympiad
Croatia
Theoretical Competition
Monday, July 19th 2010

Please read this first:

1. The examination lasts for 5 hours. There are 3 questions each worth **10 points**.

2. Use only the pen issued on your table.

3. Use only the front side of the sheets of paper provided.

4. Use the **Answer Sheets** provided to fill in your answers. Numerical results should be written with as many digits as are appropriate to the given data. ***Do not forget to state the units*.**

5. Additional **Writing Sheets** are also provided. Write on the working sheets of paper whatever you consider is required for the solution of the questions and that you wish to be marked. However you should use ***as little text as possible*** and provide only equations, numbers, symbols and diagrams.

6. It is absolutely essential that you enter your **Country Code** and your **Student Code** in the boxes at the top of every sheet of paper used. In addition, on the working sheets of paper used for each question, you should enter the number of the problem (**Problem No.**), the task number (**Task No.**), the progressive number of each sheet (**Page No.**) and the total number of working sheets that you have used and wish to be marked for each question (**Total No. of pages**). If you use some working sheets of paper for notes that you do not wish to be marked, put a large cross through the whole sheet and do not include it in your numbering.

7. When you have finished, arrange all sheets in *proper order*. For each question put

   - answer sheets first;
   - working sheets in order;
   - the sheets you do not wish to be marked;
   - put unused sheets and the printed question at the bottom.

   Place the papers for each question in the order of questions and page numbers. Bind them with the paper clip provided and leave everything on your desk. ***You are not allowed to take any sheet of paper out of the room.***

1. Image of a charge in a metallic object

Introduction – Method of images

A point charge q is placed in the vicinity of a grounded metallic sphere of radius R [see Fig. 1(a)], and consequently a surface charge distribution is induced on the sphere. To calculate the electric field and potential from the distribution of the surface charge is a formidable task. However, the calculation can be considerably simplified by using the so called method of images. In this method, the electric field and potential produced by the charge distributed on the sphere can be represented as an electric field and potential of a single point charge q' placed inside the sphere (you do not have to prove it). Note: **The electric field of this image charge q' reproduces the electric field and the potential only outside the sphere (including its surface)**.

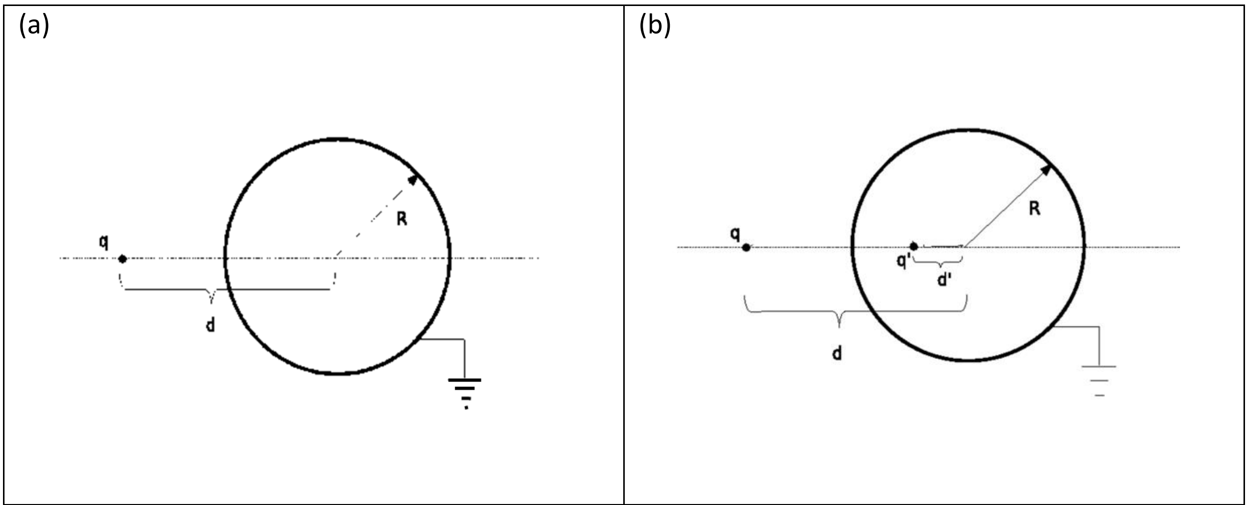


Fig 1. (a) A point charge q in the vicinity of a grounded metallic sphere. (b) The electric field of the charge induced on the sphere can be represented as electric field of an image charge q'.

Task 1 – The image charge

The symmetry of the problem dictates that the charge q' should be placed on the line connecting the point charge q and the center of the sphere [see Fig. 1(b)].

a) What is the value of the potential on the sphere? *(0.3 points)*

b) Express q' and the distance d' of the charge q' from the center of the sphere, in terms of q, d, and R. *(1.9 points)*

c) Find the magnitude of force acting on charge q. Is the force repulsive? *(0.5 points)*

Task 2 – Shielding of an electrostatic field

Consider a point charge *q* placed at a distance *d* from the center of a grounded metallic sphere of radius *R*. We are interested in how the grounded metallic sphere affects the electric field at point *A* on the opposite side of the sphere (see Fig. 2). Point *A* is on the line connecting charge *q* and the center of the sphere; its distance from the point charge *q* is *r*.

a) Find the vector of the electric field at point *A*. *(0.6 points)*

b) For a very large distance $r >> d$, find the expression for the electric field by using the approximation $(1+a)^{-2} \approx 1-2a$, where $a << 1$. *(0.6 points)*
c) In which limit of *d* does the grounded metallic sphere screen the field of the charge *q* completely, such that the electric field at point *A* is exactly zero? *(0.3 points)*

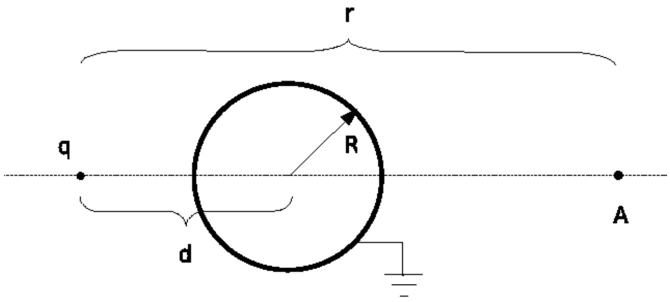


Fig 2. The electric field at point *A* is partially screened by the grounded sphere.

Task 3 – Small oscillations in the electric field of the grounded metallic sphere

A point charge *q* with mass *m* is suspended on a thread of length *L* which is attached to a wall, in the vicinity of the grounded metallic sphere. In your considerations, ignore all electrostatic effects of the wall. The point charge makes a mathematical pendulum (see Fig. 3). The point at which the thread is attached to the wall is at a distance l from the center of the sphere. Assume that the effects of gravity are negligible.

a) Find the magnitude of the electric force acting on the point charge *q* for a given angle α and indicate the direction in a clear diagram *(0.8 points)*

b) Determine the component of this force acting in the direction perpendicular to the thread in terms of l, *L, R, q and* α. *(0.8 points)*

c) Find the frequency for small oscillations of the pendulum. *(1.0 points)*

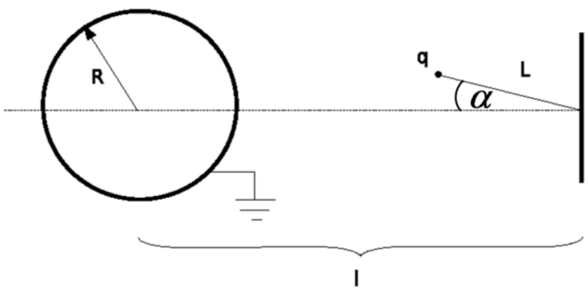


Fig 3. A point charge in the vicinity of a grounded sphere oscillates as a pendulum.

Task 4 – The electrostatic energy of the system

For a distribution of electric charges it is important to know the electrostatic energy of the system. In our problem (see Fig. 1a), there is an electrostatic interaction between the external charge *q* and the induced charges on the sphere, and there is an electrostatic interaction among the induced charges on the sphere themselves. In terms of the charge q, radius of the sphere R and the distance d determine the following electrostatic energies:

a) the electrostatic energy of the interaction between charge q and the induced charges on the sphere; *(1.0 points)*
b) the electrostatic energy of the interaction among the induced charges on the sphere; *(1.2 points)*
c) the total electrostatic energy of the interaction in the system. *(1.0 points)*

Hint: There are several ways of solving this problem:

(1) In one of them, you can use the following integral,

$$\int_d^\infty \frac{xdx}{\left(x^2-R^2\right)^2} = \frac{1}{2}\frac{1}{d^2-R^2}.$$

(2) In another one, you can use the fact that for a collection of N charges q_i located at points $\vec{r}_i, i=1,\ldots,N$, the electrostatic energy is a sum over all pairs of charges: $V = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{\left|\vec{r}_i - \vec{r}_j\right|}.$

Image of a Charge – Answer sheets

Country code	Student code

Important: leave the points fields empty for markers!

Task 1		Points
a)		
b)		
c)		
	Yes **No**	

Answer sheets – Theoretical problem 1 – Image of a charge

Country code	Student code

Task 2		Points
a)		
b)		
c)		
Task 3		**Points**
a)		
b)		
c)		

Country code	Student code

Task 4		Points
a)		
b)		
c)		
Total:		

2. Chimney physics

Introduction

Gaseous products of burning are released into the atmosphere of temperature T_{Air} through a high chimney of cross-section *A* and height *h* (see Fig. 1). The solid matter is burned in the furnace which is at temperature T_{Smoke}. The volume of gases produced per unit time in the furnace is *B*.

Assume that:

- the velocity of the gases in the furnace is negligibly small
- the density of the gases (smoke) does not differ from that of the air at the same temperature and pressure; while in furnace, the gases can be treated as ideal
- the pressure of the air changes with height in accordance with the hydrostatic law; the change of the density of the air with height is negligible
- the flow of gases fulfills the Bernoulli equation which states that the following quantity is conserved in all points of the flow:
  $\frac{1}{2}\rho v^2(z) + \rho g z + p(z) = const,$
  where ρ is the density of the gas, $v(z)$ is its velocity, $p(z)$ is pressure, and z is the height
- the change of the density of the gas is negligible throughout the chimney

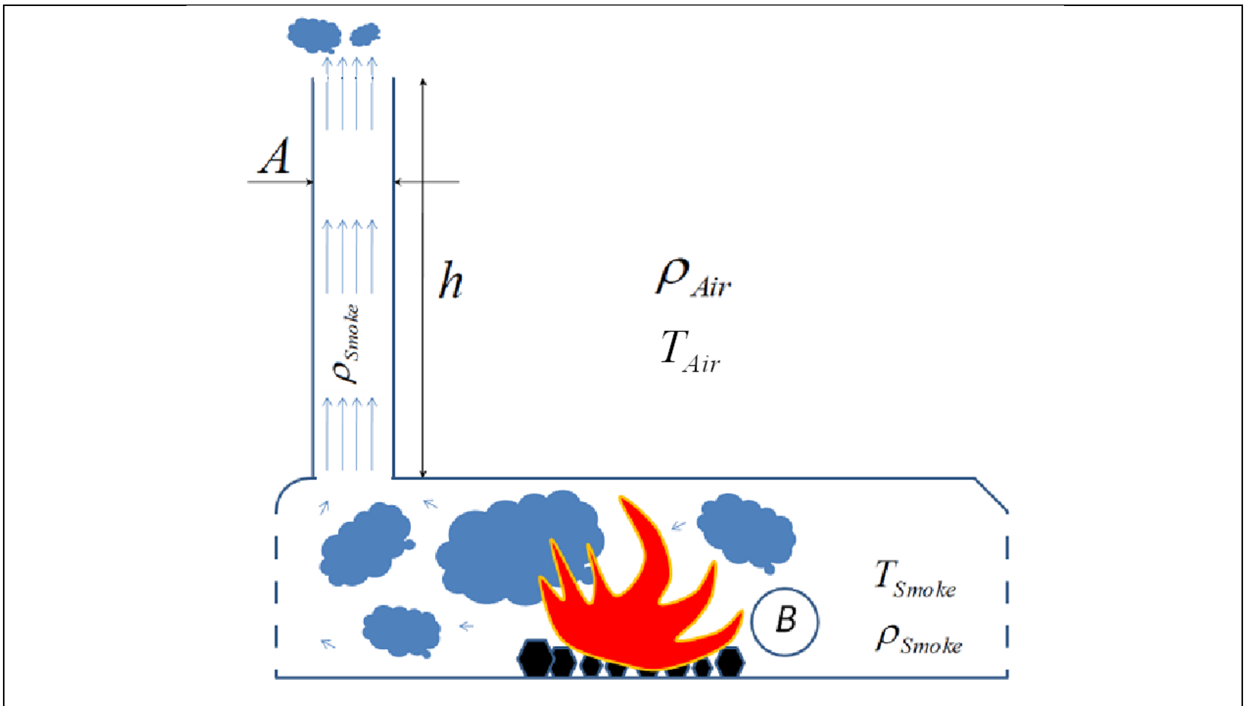


Fig 1. Sketch of a chimney of height *h* with a furnace at temperature T_{Smoke} .

Task 1

a) What is the minimal height of the chimney needed in order that the chimney functions efficiently, so that it can release all of the produced gas into the atmosphere? Express your result in terms of B, A, T_{Air}, g=9.81m/s^2, $\Delta T=T_{Smoke}-T_{Air}$. **Important: in all subsequent tasks assume that this minimal height is *the* height of the chimney.** *(3.5 points)*

b) Assume that two chimneys are built to serve exactly the same purpose. Their cross sections are identical, but are designed to work in different parts of the world: one in cold regions, designed to work at an average atmospheric temperature of -30 ^{o}C and the other in warm regions, designed to work at an average atmospheric temperature of 30 ^{o}C. The temperature of the furnace is 400 ^{o}C. It was calculated that the height of the chimney designed to work in cold regions is 100 m. How high is the other chimney? *(0.5 points)*

c) How does the velocity of the gases vary along the height of the chimney? Make a sketch/diagram assuming that the chimney cross-section does not change along the height. Indicate the point where the gases enter the chimney. *(0.6 points)*

d) How does the pressure of the gases vary along the height of the chimney? *(0.5 points)*

Solar power plant

The flow of gases in a chimney can be used to construct a particular kind of solar power plant (solar chimney). The idea is illustrated in Fig. 2. The Sun heats the air underneath the collector of area S with an open periphery to allow the undisturbed inflow of air (see Fig. 2). As the heated air rises through the chimney (thin solid arrows), new cold air enters the collector from its surrounding (thick dotted arrows) enabling a continuous flow of air through the power plant. The flow of air through the chimney powers a turbine, resulting in the production of electrical energy. The energy of solar radiation per unit time per unit of horizontal area of the collector is G. Assume that all that energy can be used to heat the air in the collector (the mass heat capacity of the air is c, and one can neglect its dependence on the air temperature). We define the efficiency of the solar chimney as the ratio of the kinetic energy of the gas flow and the solar energy absorbed in heating of the air prior to its entry into the chimney.

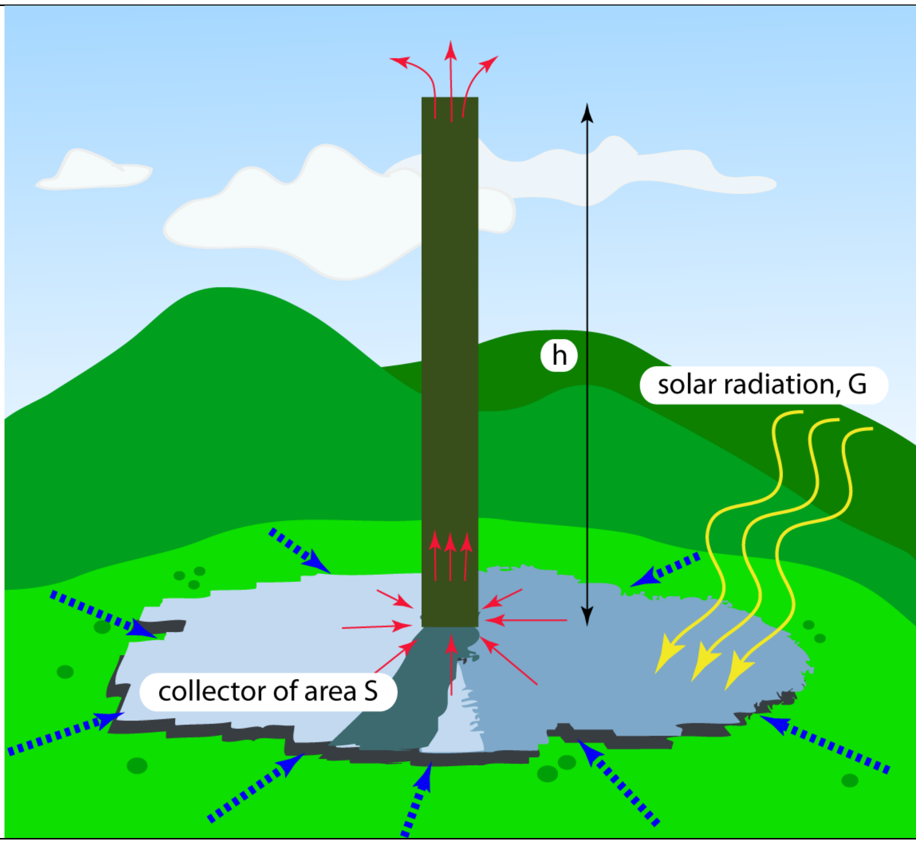


Fig 2. Sketch of a solar power plant.

Task 2

a) What is the efficiency of the solar chimney power plant? *(2.0 points)*

b) Make a diagram showing how the efficiency of the chimney changes with its height. *(0.4 points)*

Manzanares prototype

The prototype chimney built in Manzanares, Spain, had a height of 195 m, and a radius 5 m. The collector is circular with diameter of 244 m. The specific heat of the air under typical operational conditions of the prototype solar chimney is 1012 J/kg K, the density of the hot air is about 0.9 kg/m^3, and the typical temperature of the atmosphere T_{Air} = 295 K. In Manzanares, the solar power per unit of horizontal surface is typically 150 W/m^2 during a sunny day.

Task 3

a) What is the efficiency of the prototype power plant? Write down the numerical estimate. *(0.3 points)*

b) How much power could be produced in the prototype power plant? *(0.4 points)*

c) How much energy could the power plant produce during a typical sunny day? *(0.3 points)*

Task 4

a) How large is the rise in the air temperature as it enters the chimney (warm air) from the surrounding (cold air)? Write the general formula and evaluate it for the prototype chimney. *(1.0 points)*

b) What is the mass flow rate of air through the system? *(0.5 points)*

Chimney physics – Answer sheets

Country code	Student code

Important: leave the points fields empty for markers!

Task 1		Points
a)		
b)		
c)		
d)		

Answer sheets – Theoretical problem 2 – Chimney physics

Country code	Student code

Task 2		Points
a)		
b)		

Task 3		Points
a)		
b)		
c)		

Country code	Student code

Task 4		Points
a)		
b)		
Total:		

3. Simple model of an atomic nucleus

Introduction

Although atomic nuclei are quantum objects, a number of phenomenological laws for their basic properties (like radius or binding energy) can be deduced from simple assumptions: (i) nuclei are built from nucleons (i.e. protons and neutrons); (ii) strong nuclear interaction holding these nucleons together has a very short range (it acts only between neighboring nucleons); (iii) the number of protons (Z) in a given nucleus is approximately equal to the number of neutrons (N), i.e. $Z \approx N \approx A/2$, where A is the total number of nucleons ($A \gg 1$). **Important: Use these assumptions in Tasks 1-4 below.**

Task 1 - Atomic nucleus as closely packed system of nucleons

In a simple model, an atomic nucleus can be thought of as a ball consisting of closely packed nucleons [see Fig. 1(a)], where the nucleons are hard balls of radius $r_N = 0.85$ fm (1 fm = 10^{-15} m). The nuclear force is present only for two nucleons in contact. The volume of the nucleus V is larger than the volume of all nucleons AV_N, where $V_N = \frac{4}{3} r_N^3 \pi$. The ratio $f = AV_N / V$ is called the packing factor and gives the percentage of space filled by the nuclear matter.

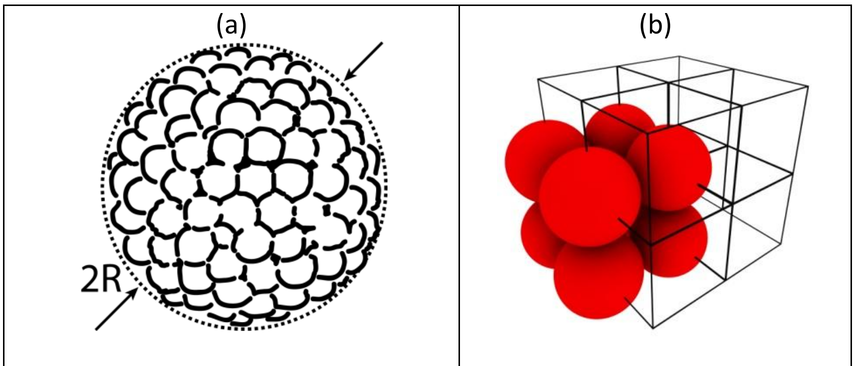


Fig. 1. (a) An atomic nucleus as a ball of closely packed nucleons. (b) The SC packing.

a) Calculate what would be the packing factor f if nucleons were arranged in a "simple cubic" (SC) crystal system, where each nucleon is centered on a lattice point of an infinite cubic lattice [see Fig. 1(b)]. *(0.3 points)*

Important: In all subsequent tasks, assume that the actual packing factor for nuclei is equal to the one from Task 1a. If you are not able to calculate it, in subsequent tasks use $f = 1/2$.

b) Estimate the average mass density ρ_m, charge density ρ_c, and the radius R for a nucleus having A nucleons. The average mass of a nucleon is $1.67 \cdot 10^{-27}$ kg. *(1.0 points)*

Task 2 - Binding energy of atomic nuclei – volume and surface terms

Binding energy of a nucleus is the energy required to disassemble it into separate nucleons and it essentially comes from the attractive nuclear force of each nucleon with its neighbors. If a given nucleon is not on the surface of the nucleus, it contributes to the total binding energy with a_V= 15.8 MeV (1 MeV = $1.602\cdot 10^{-13}$ J). The contribution of one surface nucleon to the binding energy is approximately $a_V/2$. Express the binding energy E_b of a nucleus with A nucleons in terms of A, a_V, and f, and by including the surface correction. *(1.9 points)*

Task 3 - Electrostatic (Coulomb) effects on the binding energy

The electrostatic energy of a homogeneously charged ball (with radius R and total charge Q_0) is $U_c = \dfrac{3Q_0^2}{20\pi\varepsilon_0 R}$, where $\varepsilon_0 = 8.85\cdot 10^{-12}\ \mathrm{C}^2 N^{-1} m^{-2}$.

a) Apply this formula to get the electrostatic energy of a nucleus. In a nucleus, each proton is not acting upon itself (by Coulomb force), but only upon the rest of the protons. One can take this into account by replacing $Z^2 \to Z(Z-1)$ in the obtained formula. Use this correction in subsequent tasks. *(0.4 points)*

b) Write down the complete formula for binding energy, including the main (volume) term, the surface correction term and the obtained electrostatic correction. *(0.3 points)*

Task 4 - Fission of heavy nuclei

Fission is a nuclear process in which a nucleus splits into smaller parts (lighter nuclei). Suppose that a nucleus with A nucleons splits into only two equal parts as depicted in Fig. 2.

a) Calculate the total kinetic energy of the fission products E_{kin} when the centers of two lighter nuclei are separated by the distance $d \geq 2R(A/2)$, where $R(A/2)$ is their radius. The large nucleus was initially at rest. *(1.3 points)*

b) Assume that $d = 2R(A/2)$ and evaluate the expression for E_{kin} obtained in part a) for A= 100, 150, 200 and 250 (express the results in units of MeV). Estimate the values of A for which fission is possible in the model described above? *(1.0 points)*

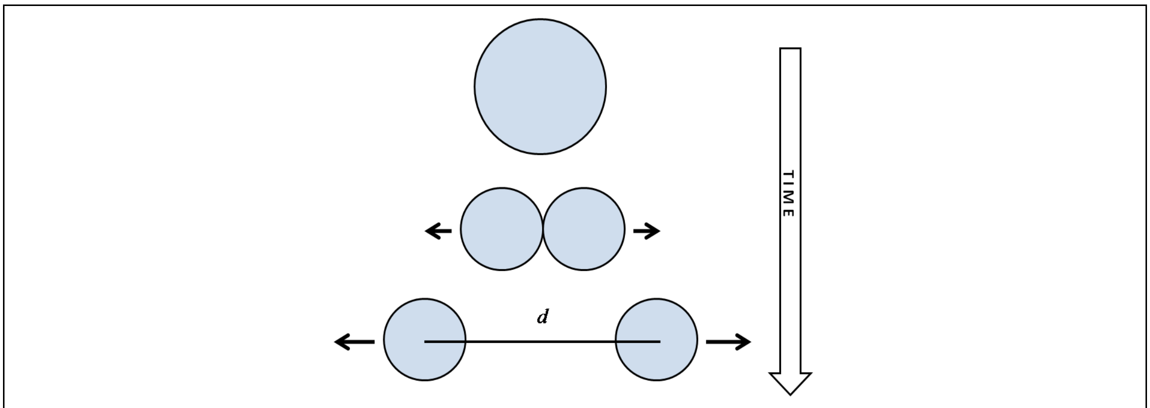


Fig. 2. A schematic description of nuclear fission in our model.

Task 5 – Transfer reactions

a) In modern physics, the energetics of nuclei and their reactions is described in terms of masses. For example, if a nucleus (with zero velocity) is in an excited state with energy E_{exc} above the ground state, its mass is $m = m_0 + E_{exc} / c^2$, where m_0 is its mass in the ground state at rest. The nuclear reaction ^{16}O+^{54}Fe→^{12}C+^{58}Ni is an example of the so-called "transfer reactions", in which a part of one nucleus ("cluster") is transferred to the other (see Fig. 3). In our example the transferred part is a ^{4}He-cluster (α -particle). The transfer reactions occur with maximum probability if the velocity of the projectile-like reaction product (in our case: ^{12}C) is equal both in magnitude and direction to the velocity of projectile (in our case: ^{16}O). The target ^{54}Fe is initially at rest. In the reaction, ^{58}Ni is excited into one of its higher-lying states. Find the excitation energy of that state (and express it units of MeV) if the kinetic energy of the projectile ^{16}O is 50 MeV. The speed of light is c= 3·10^{8} m/s. *(2.2 points)*

1.	M(^{16}O)	15.99491 a.m.u.
2.	M(^{54}Fe)	53.93962 a.m.u.
3.	M(^{12}C)	12.00000 a.m.u.
4.	M(^{58}Ni)	57.93535 a.m.u.

Table 1. The rest masses of the reactants in their ground states. 1 a.m.u.= 1.6605·10^{-27} kg.

b) The ^{58}Ni nucleus produced in the excited state discussed in the part a), deexcites into its ground state by emitting a gamma-photon in the direction of its motion. Consider this decay in the frame of reference in which ^{58}Ni is at rest to find the recoil energy of ^{58}Ni (*i.e.* kinetic energy which ^{58}Ni acquires after the emission of the photon). What is the photon energy in that system? What is the photon energy in the lab system of reference (i.e. what would be the energy of the photon measured in the detector which is positioned in the direction in which the ^{58}Ni nucleus moves)? *(1.6 points)*

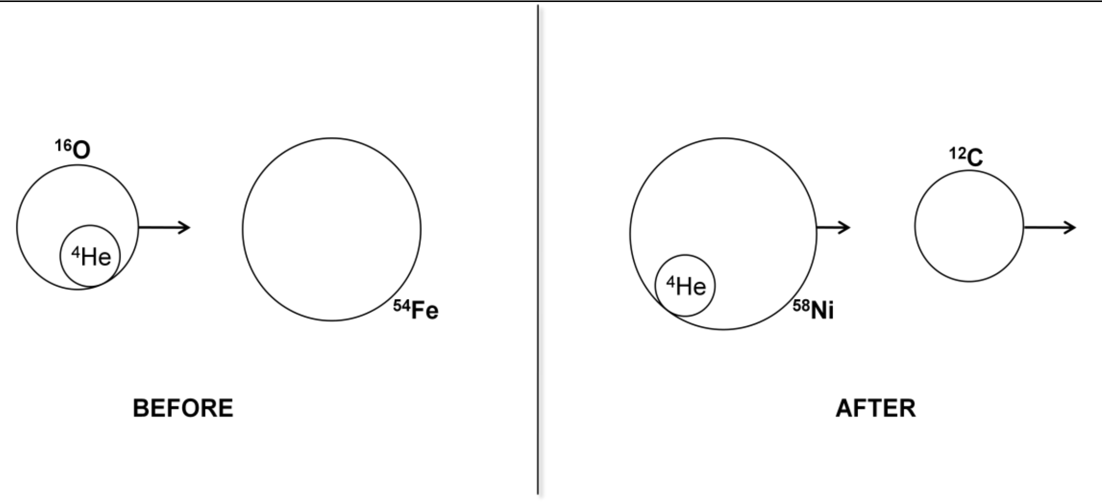


Fig. 3. The schematics of a transfer reaction.

Simple model of atomic nucleus – Answer sheets

Country code	Student code

Important: leave the Points fields empty for markers!

Task 1		Points
a)		
b)	$\rho_m =$	
	$\rho_c =$	
	$R =$	
Task 2		**Points**

Answer sheets – Theoretical problem 3 – Nuclear model

Country code	Student code

Task 3		Points
a)		
b)		

Task 4		Points
a)		
b)	$E_{kin}(A=100)=$ $E_{kin}(A=150)=$ $E_{kin}(A=200)=$ $E_{kin}(A=250)=$ necessary condition for fission:	

Country code	Student code

Task 5		Points
a)		
b)	$E_\gamma =$	
	$E_{\text{recoil}} =$	
	$E_{\text{detector}} =$	
Total:		

The 41st International Physics Olympiad
Croatia
Experimental Competition
Wednesday, July 21st, 2010

Please read this first:

1. The time given is 5 hours.
2. There are two experimental problems. Each experiment is awarded 10 points.
3. Use only the provided setup, pencil, and sheets.
4. Write your solutions in the **Answer sheets**. **Working sheets** can be used if necessary. All *will be considered for marks*.
5. When using working sheets:
   - Use only the front side of the paper. Start each part on a fresh sheet of paper.
   - On every paper, write:
     1) the **Task No.** for the task attempted
     2) the **Page No.** - the progressive number of each sheet for that part
     3) the **Total No. of Pages** used for that part
     4) your **Country Code** and your **Student Code**
   - Write concisely – Limit the use of text to minimum. Use equations, numbers, symbols, figures and graphs as far as possible.
   - Cross out pages that you do not wish to be marked. Do not include them in your numbering.
6. For each task, use the **Answer Sheet** to fill in your *final answer* in the appropriate box. Give the appropriate number of significant figures. Remember to state the units.
7. When you have finished, arrange all sheets in this order *for each part*:
   - the **Answer Sheet**
   - writing sheets that you wish to be marked
   - writing sheets that you do not wish to be marked

   Place all unused sheets, graph papers and the question paper at the bottom.
8. Clip *all sheets* together and leave them on your desk.
9. You are not allowed to take *any* sheet of paper or *any* material used in the experiment out of the examination hall.

Separate instructions for using the scale

The scale is turned **ON-OFF** by the **right button**.
The **middle button** (Z/T) sets the digits to zero, that is, this is the **TARA** function.
The **left button** can be used to change **units**.
Instruction: Put units to grams in case it is in other units!

Separate instructions for using the press

The press is used in both problems. The upper part of the press is turned up-side down in the second experiment as compared to the first. **Its position is illustrated in the tasks themselves.** The stone is to be placed on the upper part of the press. Its weight helps the upper part of the press to slide down when you turn the wing-nut (if you find necessary, you can gently press the upper part by your hand (close to the vertical bar) while you turn the wing nut to ensure smooth sliding of the press). **For performing measurements, you should use the fact that the upper part of the press moves 2 mm when the wing nut is rotated 360 degrees.**

SAFETY WARNING

You should be careful when playing with the wooden stick, the rod magnet and the hollow cylinder.

Be careful not to stick the wooden stick in your eyes!!!

Do not look with your eyes into the hollow cylinder when playing with the rod magnet inside the cylinder. It can be ejected from the cylinder and injure your eyes.

Experimental problem 1

There are two experimental problems. The setup on your table is used for both problems. You have 5 hours to complete the entire task (1&2).

Experimental problem 1: Elasticity of sheets

Introduction

Springs are objects made from elastic materials which can be used to store mechanical energy. The most famous helical springs are well described in terms of Hooke's law, which states that the force with which the spring pushes back is linearly proportional to the distance from its equilibrium length: $F = -k\Delta x$, where k is the spring constant, Δx is the displacement from equilibrium, and F is the force [see Fig. 1(a)]. However, elastic springs can have quite different shapes from the usual helical springs, and for larger deformations Hooke's law does not generally apply. In this problem we measure the properties of a spring made from a sheet of elastic material, which is schematically illustrated in Fig. 1(b).

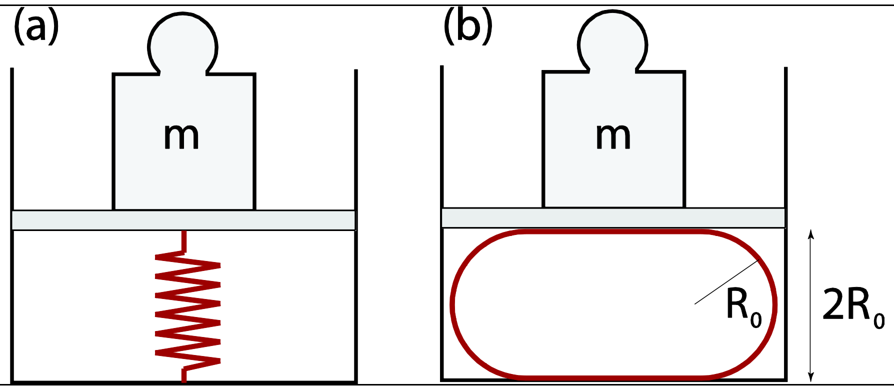


Figure 1. Illustrations of (a) a helical spring and (b) a spring made from a sheet of elastic material rolled up into a cylinder. When the latter spring is sufficiently compressed, its shape can be approximated as a stadium with two semicircles of radii R_0 (see text).

Transparent foil rolled into a cylindrical spring

Suppose that we take a sheet of elastic material (e.g. a transparent foil) and bend it. The more we bend it, the more elastic energy is stored in the sheet. The elastic energy depends on the curvature of the sheet. Parts of the sheet with larger curvature store more energy (flat parts of the sheet do not store energy because their curvature is zero). The springs used in this experiment are made from rectangular transparent foils rolled into cylinders (see Fig. 2). The elastic energy stored in a cylinder is

$E_{el} = \frac{\kappa}{2}\frac{1}{R_c^2}A,$	(1)

where A denotes the area of the cylinder's side (excluding its bases), R_c denotes its radius, and the parameter κ, referred to as the bending rigidity, is determined by the elastic properties of the material and the thickness of the sheet. Here we neglect the stretching of the sheets.

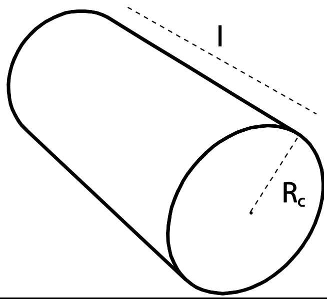


Figure 2. A schematic picture of an elastic sheet rolled into a cylinder of radius R_c and length l.

Suppose that such a cylinder is compressed as in Fig. 1(b). For a given force applied by the press (F), the displacement from equilibrium depends on the elasticity of the transparent foil. For some interval of compression forces, the shape of the compressed transparency foil can be approximated with the shape of a stadium, which has a cross section with two straight lines and two semicircles, both of radius R_0. It can be shown that the energy of the compressed system is minimal when

$$R_0^2 = \frac{l\kappa\pi}{2F}. \tag{2}$$

The force is measured by the scale calibrated to measure mass m, so $F = mg$, $g = 9.81$ m/s^2.

Experimental setup (1st problem)

The following items (to be used for the 1st problem) are on your desk:

1. Press (together with a stone block); see separate instructions if needed
2. Scale (measures mass up to 5000 g, it has TARA function, see separate instructions if needed)
3. Transparency foils (all foils are 21 cm x 29.7 cm, the blue foil is 200 µm thick, and the colorless foil is 150 µm thick); please, do ask for the extra foils if you need them.
4. Adhesive (scotch) tape
5. Scissors
6. Ruler with a scale
7. A rectangular wooden plate (the plate is to be placed on a scale, and the foil sits on the plate)

The setup is to be used as in Fig. 3. The upper plate of the press can be moved downward and upward using a wing nut, and the force (mass) applied by the press is measured with the scale. **Important: The wing nut moves 2 mm when rotated 360 degrees.** (Small aluminum rod is not used in Experiment 1.)

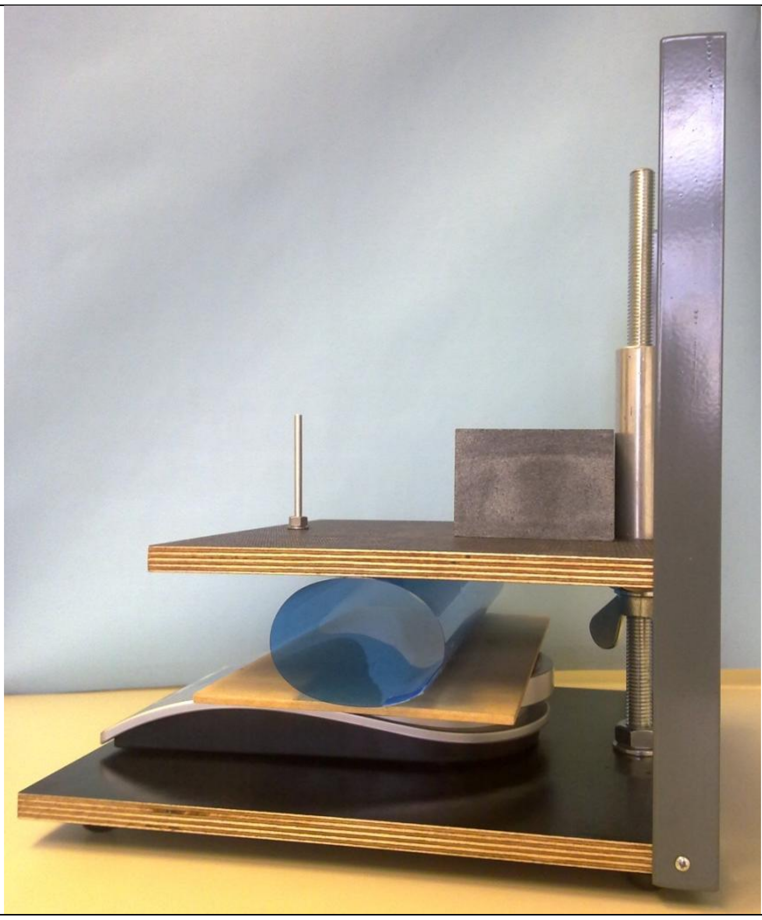


Figure 3. The photo of a setup for measuring the bending rigidity.

Tasks

1. Roll the blue foils into cylinders, one along the longer side, and the other along its shorter side; use the adhesive tape to fix them. The overlap of the sheet should be about 0.5 cm.
   (a) Measure the dependence of the mass read by the scale on the separation between the plates of the press for each of the two cylinders. *(1.9 points)*
   (b) Plot your measurements on appropriate graphs. Using the ruler and eye as the guide, draw lines through the points and determine the bending rigidities κ for the cylinders. Mark the region where the approximate relation (the stadium approximation) holds. Estimate the value of $\frac{R_0}{R_c}$ below which the stadium approximation holds; here R_c is the radius of the non-laden cylinder(s). *(4.3 points)*

   The error analysis of the results is not required.

2. Measure the bending rigidity of a single colorless transparent foil. *(2.8 points)*
3. The bending rigidity κ depends on the Young's modulus Y of elasticity of the isotropic material, and the thickness d of the transparent foil according to

$\kappa = \dfrac{Yd^3}{12(1-\nu^2)}$,	(3)

where ν is the Poisson ratio for the material; for most materials $\nu \approx 1/3$. From the previous measurements, determine the Young's modulus of the blue and the colorless transparent foil. *(1.0 points)*

Exp. problem 1 – Answer sheets

Country code	Student code

Task 1		Points
(a)		

Country code	Student code

Task 1		Points
(a)		

Country code	Student code

Task 1		Points
(b)		

Country code	Student code

Task 1		Points
(b)		

Country code	**Student code**

Task 1		Points
(b)	$\kappa =$	
	$\kappa =$	
	$\frac{R_0}{R_c} \leq$ $\frac{R_0}{R_c} \leq$	
Task 2		**Points**

Country code	Student code

Task 2	Points

Country code	Student code

Task 2		Points
	$\kappa =$	
Task 3		**Points**
	Young modulus of the blue foil:	
	Young modulus of the colorless foil:	
Total:		

Experimental problem 2

There are two experimental problems. The setup on your table is used for both problems. You have 5 hours to complete the entire task (1&2).

Experimental problem 2: Forces between magnets, concepts of stability and symmetry

Introduction

Electric current I circulating in a loop of area S creates a magnetic moment of magnitude $m = IS$ [see Fig. 1(a)]. A permanent magnet can be thought of as a collection of small magnetic moments of iron (Fe), each of which is analogous to the magnetic moment of a current loop. This (Ampère's) model of a magnet is illustrated in Fig. 1(b). The total magnetic moment is a sum of all small magnetic moments, and it points from the south to the northern pole.

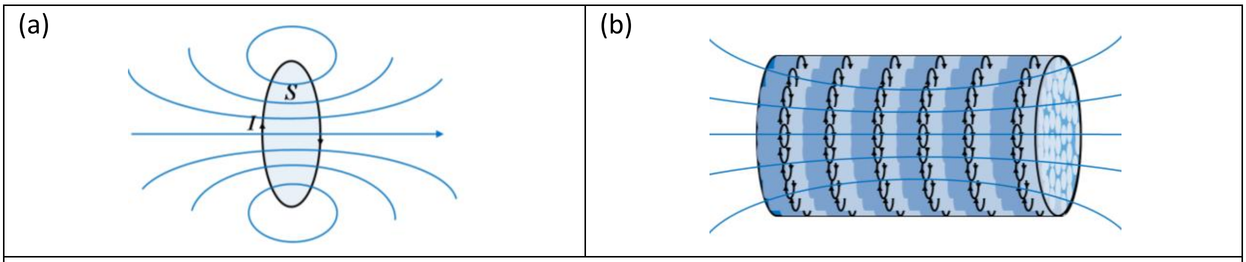


Figure 1. (a) Illustration of a current loop and the produced magnetic field. (b) Ampère's model of a magnet as a collection of small current loops.

Forces between magnets

To calculate the force between two magnets is a nontrivial theoretical task. It is known that like poles of two magnets repel, and unlike poles attract. The force between two current loops depends on the strengths of the currents in them, their shape, and their mutual distance. If we reverse the current in one of the loops, the force between them will be of the same magnitude, but of the opposite direction.

In this problem you experimentally investigate the forces between two magnets, the ring-magnet and the rod-magnet. We are interested in the geometry where the axes of symmetry of the two magnets coincide (see Fig. 2). The rod-magnet can move along the z - axis from the left, through the ring-magnet, and then towards the right (see Fig. 2). Among other tasks, you will be asked to measure the force between the magnets as a function of z. The origin $z = 0$ corresponds to the case when the centers of the magnets coincide.

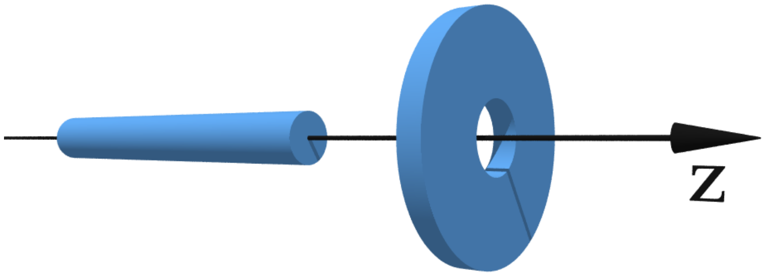


Figure 2. The rod- and the ring-magnet are aligned. The force between them changes as the rod-magnet moves along the z - axis.

To ensure motion of the rod-magnet along the axis of symmetry (z - axis), the ring-magnet is firmly embedded in a transparent cylinder, which has a narrow hole drilled along the z - axis. The rod-magnet is thus constrained to move along the z - axis through the hole (see Fig. 3). The magnetization of the magnets is along the z - axis. The hole ensures radial stability of the magnets.

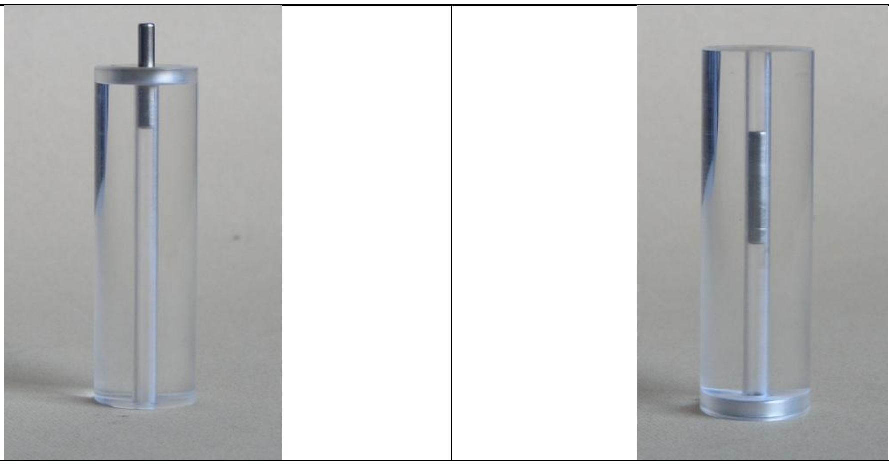


Figure 3. Photo of two magnets and a transparent hollow cylinder; the rod-magnet moves through the cylinder's hole.

Experimental setup (2nd problem)

The following items (to be used for the 2nd problem) are on your desk:

1. Press (together with a stone block); see separate instructions if needed
2. Scale (measures mass up to 5000 g, it has TARA function, see separate instructions if needed)
3. A transparent hollow cylinder with a ring-magnet embedded in its side.
4. One rod-magnet.
5. One narrow wooden stick (can be used to push the rod magnet out of the cylinder).

The setup is to be used as in Fig. 4 to measure the forces between the magnets. The upper plate of the press needs to be turned up-side-down in comparison to the first experimental problem. The narrow Aluminum rod is used to press the rod-magnet through the hollow cylinder. The scale measures the force (mass). The upper plate of the press can be moved downwards and upwards by using a wing nut. **Important: The wing nut moves 2mm when rotated 360 degrees.**

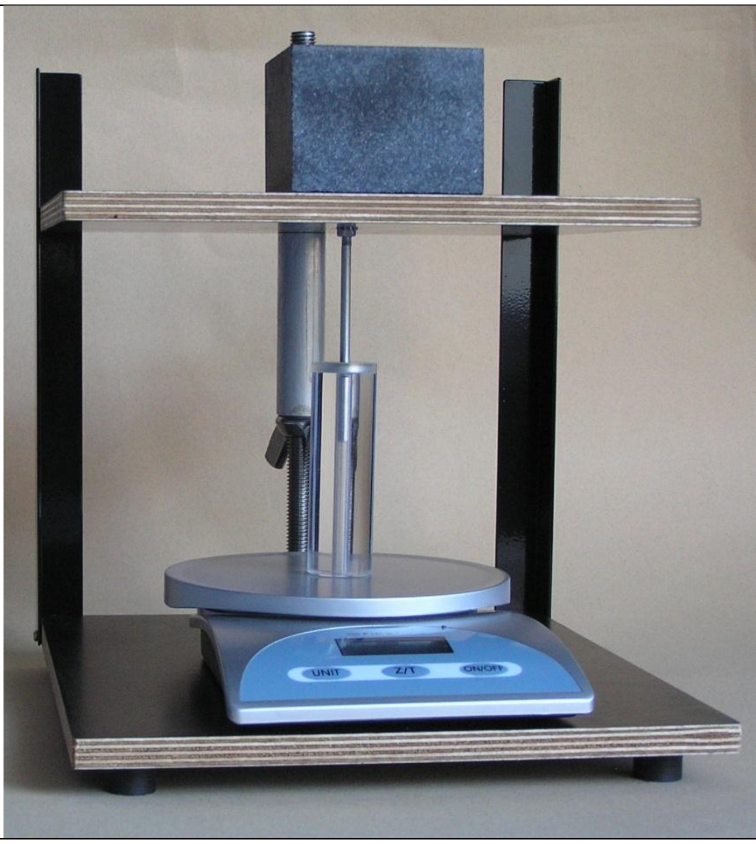


Figure 4. Photograph of the setup, and the way it should be used for measuring the force between the magnets.

Tasks

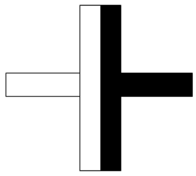
1. Determine qualitatively all equilibrium positions between the two magnets, assuming that the z - axis is positioned horizontally as in Fig. 2 , and draw them in the answer sheet. Label the equilibrium positions as stable (S)/unstable (U), and denote the like poles by shading, as indicated for one stable position in the answer sheet. You can do this Task by using your hands and a wooden stick. *(2.5 points)*
2. By using the experimental setup measure the force between the two magnets as a function of the z - coordinate. Let the positive direction of the z - axis point into the transparent cylinder (the force is positive if it points in the positive direction). For the configuration when the magnetic moments are parallel, denote the magnetic force by $F_{\uparrow\uparrow}(z)$, and when they are anti-parallel, denote the magnetic force by $F_{\uparrow\downarrow}(z)$. **Important: Neglect the mass of the rod-magnet (i.e., neglect gravity), and utilize the symmetries of the forces between magnets to measure different parts of the curves.** If you find any symmetry in the forces, write them in the answer sheets. Write the measurements on the answer sheets; beside every table schematically draw the configuration of magnets corresponding to each table (an example is given). *(3.0 points)*
3. By using the measurements from Task 2, use the millimeter paper to plot in detail the functional dependence $F_{\uparrow\uparrow}(z)$ for $z > 0$. Plot schematically the shapes of the curves $F_{\uparrow\uparrow}(z)$ and $F_{\uparrow\downarrow}(z)$ (along the positive and the negative z - axis). On each schematic graph

denote the positions of the stable equilibrium points, and sketch the corresponding configuration of magnets (as in Task 1). *(4.0 points)*

4. If we do not neglect the mass of the rod magnet, are there any qualitatively new stable equilibrium positions created when the z - axis is positioned vertically? If so, plot them on the answer sheet as in Task 1. *(0.5 points)*

Answer Sheets - Exp. Problem 2

Country code	Student code

Task 1		Points
	S U 	
	S U	
	S U	
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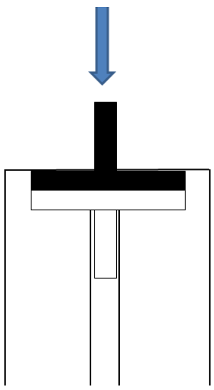
	S U	

Country code	Student code

Task 1		Points
	S U	
	S U	
	S U	
	S U	
	S U	

Country code	Student code

Task 2		Points
	Write all symmetries that you find for the force between magnets:	
	Configuration:  Measurements:	

Country code	Student code

Task 2		Points

Country code	Student code

Task 2		Points

Country code	Student code

Task 3	Points

Country code	Student code

Task 3		Points
	Schematic (hand drawn) plot of $F_{\uparrow\uparrow}(z)$	
	Schematic (hand drawn) plot of $F_{\uparrow\downarrow}(z)$	

Country code	Student code

Task 4		Points
Total:		

Solution - Image of a charge

Solution of Task 1

Task 1a)

As the metallic sphere is grounded, its potential vanishes, V=0.

Task1b)

Let us consider an arbitrary point B on the surface of the sphere as depicted in Fig. 1.

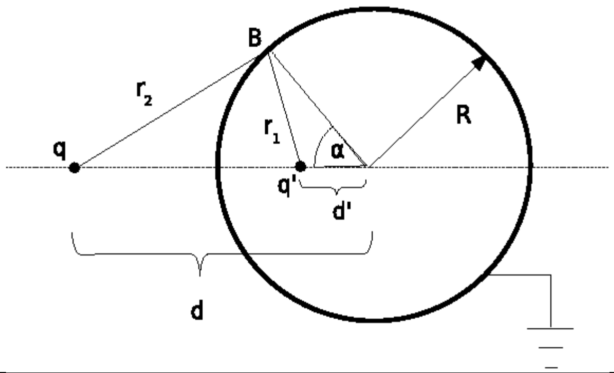


Fig 1. The potential at point *B* is zero.

The distance of point *B* from the charge q' is

$$r_1 = \sqrt{R^2 + d'^2 - 2Rd' \cos\alpha} \quad (1)$$

whereas the distance of the point B from the charge q is given with the expression

$$r_2 = \sqrt{R^2 + d^2 - 2Rd \cos\alpha} \quad (2)$$

The electric potential at the point B is

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_2} + \frac{q'}{r_1}\right) \quad (3)$$

This potential must vanish,

$$\frac{q}{r_2} + \frac{q'}{r_1} = 0 \quad (4)$$

i.e. its numerical value is 0 V.

Combining (1), (2) and (3) we obtain

$$R^2 + d^2 - 2Rd\cos\alpha = \left(\frac{q}{q'}\right)^2 (R^2 + d'^2 - 2Rd'\cos\alpha)$$	(5)

As the surface of the sphere must be equipotential, the condition (5) must be satisfied for every angle α what leads to the following results

$$d^2 + R^2 = \left(\frac{q}{q'}\right)^2 (R^2 + d'^2)$$	(6)

and

$$dR = \left(\frac{q}{q'}\right)^2 (d'R)$$	(7)

By solving of (6) and (7) we obtain the expression for the distance d' of the charge q' from the center of the sphere

$$d' = \frac{R^2}{d}$$	(8)

and the size of the charge q'

$$q' = -q\frac{R}{d}$$	(9)

Task 1c)

Finally, the magnitude of force acting on the charge q is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2 Rd}{\left(d^2 - R^2\right)^2}$$	(10)

The force is apparently **attractive.**

Solution of Task 2

Task 2a)

The electric field at the point A amounts to

$$\vec{E}_A = \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} - \frac{1}{4\pi\varepsilon_0} \frac{q\frac{R}{d}}{\left(r - d + \frac{R^2}{d}\right)^2} \right) \hat{r} \tag{11}$$

Task 2b)

For very large distances r we can apply approximate formula $(1+a)^{-2} \approx 1-2a$ to the expression (11) what leads us to

$$\vec{E}_A = \frac{1}{4\pi\varepsilon_0} \frac{\left(1 - \frac{R}{d}\right) q}{r^2} \hat{r} - \frac{1}{4\pi\varepsilon_0} \frac{2q\frac{R}{d}\left(d - \frac{R^2}{d}\right)}{r^3} \hat{r} \tag{12}$$

In general a grounded metallic sphere cannot completely screen a point charge q at a distance d (even in the sense that its electric field would decrease with distance faster than $1/r^2$) and the dominant dependence of the electric field on the distance r is as in standard Coulomb law.

Task 2c)

In the limit $d \to R$ the electric field at the point A vanishes and the grounded metallic sphere screens the point charge completely.

Solution of Task 3

Task 3a)

Let us consider a configuration as in Fig. 2.

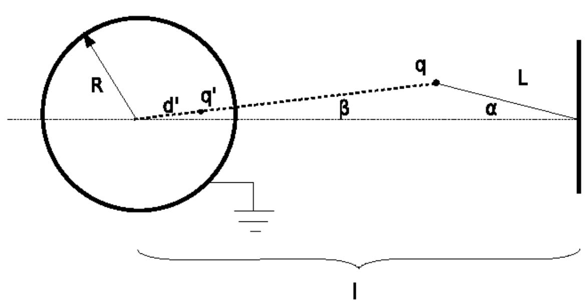


Fig 2. The pendulum formed by a charge near a grounded metallic sphere.

The distance of the charge q from the center of the sphere is

$$d = \sqrt{l^2 + L^2 - 2lL\cos\alpha}$$	(13)

The magnitude of the electric force acting on the charge q is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(d-d')^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2Rd}{(d^2-R^2)^2}$$	(14)

From which we have

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2R\sqrt{l^2+L^2-2lL\cos\alpha}}{\left(l^2+L^2-2lL\cos\alpha-R^2\right)^2}$$	(15)

Task 3b)

The direction of the vector of the electric force (17) is described in Fig. 3.

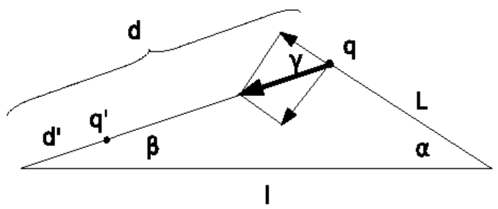


Fig 3. The direction of the force *F*.

The angles α and β are related as

$$L\sin\alpha = d\sin\beta$$	(16)

whereas for the angle γ the relation γ=α+β is valid. The component of the force perpendicular to the thread is F sin γ, that is ,

$$F_\perp = \frac{1}{4\pi\varepsilon_0}\frac{q^2R\sqrt{l^2+L^2-2lL\cos\alpha}}{\left(l^2+L^2-2lL\cos\alpha-R^2\right)^2}\sin(\alpha+\beta)$$ where $$\beta = \arcsin\left(\frac{L}{\sqrt{L^2+l^2-2Ll\cos\alpha}}\sin\alpha\right)$$	(17)

Task 3c)

The equation of motion of the mathematical pendulum is

$mL\ddot{\alpha} = -F_{\perp}$	(18)

As we are interested in small oscillations, the angle α is small, i.e. for its value in radians we have α much smaller than 1. For a small value of argument of trigonometric functions we have approximate relations sin x ≈ x and cos x ≈ $1-x^2/2$. So for small oscillations of the pendulum we have $\beta \approx \alpha L/(l-L)$ and $\gamma \approx l\alpha/(l-L)$.

Combining these relations with (13) we obtain

$mL\frac{d^2\alpha}{dt^2} + \frac{1}{4\pi\varepsilon_0}\frac{q^2 Rd}{\left(d^2-R^2\right)^2}\left(1+\frac{L}{d}\right)\alpha = 0$	(19)

Where $d = l - L$ what leads to

$\omega = \frac{q}{d^2-R^2}\sqrt{\frac{Rd}{4\pi\varepsilon_0}\frac{1}{mL}\left(1+\frac{L}{d}\right)} = \frac{q}{(l-L)^2-R^2}\sqrt{\frac{Rl}{4\pi\varepsilon_0}\frac{1}{mL}}$.	(20)

Solution of Task 4

First we present a solution based on the definition of the electrostatic energy of a collection of charges.

Task 4a)

The total energy of the system can be separated into the electrostatic energy of interaction of the external charge with the induced charges on the sphere, $E_{el,1}$, and the electrostatic energy of mutual interaction of charges on the sphere, $E_{el,2}$, i.e.

$E_{el} = E_{el,1} + E_{el,2}$	(21)

Let there be *N* charges induced on the sphere. These charges q_j are located at points $\vec{r}_j, j = 1, \ldots, N$ on the sphere. We use the definition of the image charge, i.e., the potential on the surface of the sphere from the image charge is identical to the potential arising from the induced charges:

$$\frac{q'}{\left|\vec{r}-\vec{d}'\right|}=\sum_{j=1}^{N}\frac{q_j}{\left|\vec{r}_j-\vec{r}\right|}, \tag{22}$$

where $\vec{r}$ is a vector on the sphere and $\vec{d}'$ denotes the vector position of the image charge. When $\vec{r}$ coincides with some $\vec{r}_i$, then we just have

$$\frac{q'}{\left|\vec{r}_i-\vec{d}'\right|}=\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_j}{\left|\vec{r}_j-\vec{r}_i\right|}. \tag{23}$$

From the requirement that the potential on the surface of the sphere vanishes we have

$$\frac{q'}{\left|\vec{r}-\vec{d}'\right|}+\frac{q}{\left|\vec{r}-\vec{d}\right|}=0, \tag{24}$$

where $\vec{d}$ denotes the vector position of the charge $\vec{q}$ ($\vec{r}$ is on the sphere).

For the interaction of the external charge with the induced charges on the sphere we have

$$E_{el,1}=\frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{\left|\vec{r}_i-\vec{d}\right|}=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{\left|\vec{d}'-\vec{d}\right|}=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{d-d'}=-\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2} \tag{25}$$

Here the first equality is the definition of this energy as the sum of interactions of the charge *q* with each of the induced charges on the surface of the sphere. The second equality follows from (21).

In fact, the interaction energy $E_{el,1}$ follows directly from the definition of an image charge.

Task 4b)

The energy of mutual interactions of induced charges on the surface of the sphere is given with

$E_{el,2} = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_i q_j}{\lvert\vec{r}_i-\vec{r}_j\rvert}$ $= \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}q_i\frac{q'}{\lvert\vec{r}_i-\vec{d}'\rvert} =$ $= -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}q_i\frac{q}{\lvert\vec{r}_i-\vec{d}\rvert} =$ $= -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{qq'}{\lvert\vec{d}'-\vec{d}\rvert} = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{qq'}{d-d'} = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(26)

Here the second line is obtained using (22). From the second line we obtain the third line applying (23), whereas from the third line we obtain the fourth using (22) again.

Task 4c)

Combining expressions (19) and (20) with the quantitative results for the image charge we finally obtain the total energy of electrostatic interaction

$E_{el}(d) = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(27)

An alternative solution follows from the definition of work. By knowing the integral

$\int_d^\infty \frac{xdx}{(x^2-R^2)^2} = \frac{1}{2}\frac{1}{d^2-R^2}$	(28)

We can obtain the total energy in the system by calculating the work needed to bring the charge q from infinity to the distance d from the center of the sphere:

$E_{el}(d) = -\int_\infty^d F(\vec{x})d\vec{x} = \int_d^\infty F(\vec{x})d\vec{x} =$ $= \int_d^\infty (-)\frac{1}{4\pi\varepsilon_0}\frac{q^2Rx}{(x^2-R^2)^2}dx =$ $= -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(29)

This solves Task 4c).

The electrostatic energy between the charge q and the sphere must be equal to the energy between the charges q and q' according to the definition of the image charge:

$E_{el,1}=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{(d-d')}=-\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(30)

This solves Task 4a).

From this we immediately have that the electrostatic energy among the charges on the sphere is:

$E_{el,2}=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$.	(31)

This solves Task 4b).

Solution - Chimney physics

This problem was inspired and posed by using the following two references:

- W.W. Christie, *Chimney design and theory*, D. Van Nostrand Company, New York, 1902.
- J. Schlaich, R. Bergermann, W. Schiel, G. Weinrebe, *Design of Commercial Solar Updraft Tower Systems — Utilization of Solar Induced Convective Flows for Power Generation*, Journal of Solar Energy Engineering 127, 117 (2005).

Solution of Task 1

a) What is the minimal height of the chimney needed in order that the chimney functions efficiently, so that it can release all of the produced gas in the atmosphere?

Let $p(z)$ denote the pressure of air at height z; then, according to one of the assumptions $p(z) = p(0) - \rho_{Air} gz$, where $p(0)$ is the atmospheric pressure at zero altitude.

Throughout the chimney the Bernoulli law applies, that is, we can write

$\frac{1}{2}\rho_{Smoke} v(z)^2 + \rho_{Smoke} gz + p_{Smoke}(z) = const.,$	(1)

where $p_{Smoke}(z)$ is the pressure of smoke at height z, ρ_{Smoke} is its density, and $v(z)$ denotes the velocity of smoke; here we have used the assumption that the density of smoke does not vary throughout the chimney. Now we apply this equation at two points, (i) in the furnace, that is at point $z = -\varepsilon$, where ε is a negligibly small positive number, and (ii) at the top of the chimney where $z = h$ to obtain:

$\frac{1}{2}\rho_{Smoke} v(h)^2 + \rho_{Smoke} gh + p_{Smoke}(h) \approx p_{Smoke}(-\varepsilon)$	(2)

On the right hand side we have used the assumption that the velocity of gases in the furnace is negligible (and also $-\rho_{Smoke} g\varepsilon \approx 0$).

We are interested in the minimal height at which the chimney will operate. The pressure of smoke at the top of the chimney has to be equal or larger than the pressure of air at altitude h; for minimal height of the chimney we have $p_{Smoke}(h) \approx p(h)$. In the furnace we can use $p_{Smoke}(-\varepsilon) \approx p(0)$. The Bernoulli law applied in the furnace and at the top of the chimney [Eq. (2)] now reads

$\frac{1}{2}\rho_{Smoke} v(h)^2 + \rho_{Smoke} gh + p(h) \approx p(0).$	(3)

From this we get

$v(h) = \sqrt{2gh\left(\frac{\rho_{Air}}{\rho_{Smoke}} - 1\right)}.$	(4)

The chimney will be efficient if all of its products are released in the atmosphere, i.e.,

$v(h) \geq \frac{B}{A}$,	(5)

from which we have

$h \geq \frac{B^2}{A^2} \frac{1}{2g} \frac{1}{\frac{\rho_{Air}}{\rho_{Smoke}} - 1}$.	(6)

We can treat the smoke in the furnace as an ideal gas (which is at atmospheric pressure $p(0)$ and temperature T_{Smoke}). If the air was at the same temperature and pressure it would have the same density according to our assumptions. We can use this to relate the ratio $\rho_{Air} / \rho_{Smoke}$ to T_{Smoke} / T_{Air} that is,

$\frac{\rho_{Air}}{\rho_{Smoke}} = \frac{T_{Smoke}}{T_{Air}}$, and finally	(7)
$h \geq \frac{B^2}{A^2} \frac{1}{2g} \frac{T_{Air}}{T_{Smoke} - T_{Air}} = \frac{B^2}{A^2} \frac{1}{2g} \frac{T_{Air}}{\Delta T}$.	(8)

For minimal height of the chimney we use the equality sign.

b) How high should the chimney in warm regions be?

$\frac{h(30)}{h(-30)} = \frac{\frac{T(30)}{T_{Smoke} - T(30)}}{\frac{T(-30)}{T_{Smoke} - T(-30)}}; h(30) = 145m$.	(9)

c) How does the velocity of the gases vary along the height of the chimney?

The velocity is constant,

$v = \sqrt{2gh\left(\frac{\rho_{Air}}{\rho_{Smoke}} - 1\right)} = \sqrt{2gh\left(\frac{T_{Smoke}}{T_{Air}} - 1\right)} = \sqrt{2gh\frac{\Delta T}{T_{Air}}}$.	(10)

This can be seen from the equation of continuity $Av = const.$ (ρ_{Smoke} is constant). It has a sudden jump from approximately zero velocity to this constant value when the gases enter the chimney from the furnace. In fact, since the chimney operates at minimal height this constant is equal to B, that is $v = B / A$.

d) At some height *z*, from the Bernoulli equation one gets

$p_{smoke}(z) = p(0) - (\rho_{Air} - \rho_{Smoke})gh - \rho_{Smoke} gz$.	(11)

Thus the pressure of smoke suddenly changes as it enters the chimney from the furnace and acquires velocity.

Solution of Task 2

a) The kinetic energy of the hot air released in a time interval Δt is

$E_{kin} = \frac{1}{2}(Av\Delta t\rho_{Hot})v^2 = Av\Delta t\rho_{Hot}gh\frac{\Delta T}{T_{Atm}}$,	(12)

Where the index "Hot" refer to the hot air heated by the Sun. If we denote the mass of the air that exits the chimney in unit time with $w = Av\rho_{Hot}$, then the power which corresponds to kinetic energy above is

$P_{kin} = wgh\frac{\Delta T}{T_{Air}}$.	(13)

This is the maximal power that can be obtained from the kinetic energy of the gas flow.

The Sun power used to heat the air is

$P_{Sun} = GS = wc\Delta T$.	(14)

The efficiency is evidently

$\eta = \frac{P_{kin}}{P_{Sun}} = \frac{gh}{cT_{Atm}}$.	(15)

b) The change is apparently linear.

Solution of Task 3

a) The efficiency is

$\eta = \frac{gh}{cT_{Atm}} = 0.0064 = 0.64\%$.	(16)

b) The power is

$P = GS\eta = G(D/2)^2\pi\eta = 45$ kW.	(17)

c) If there are 8 sunny hours per day we get 360kWh.

Solution of Task 4

The result can be obtained by expressing the mass flow of air w as

$w = Av\rho_{Hot} = A\sqrt{2gh\frac{\Delta T}{T_{Air}}}\rho_{Hot}$	(18)
$w = \frac{GS}{c\Delta T}$	(19)

which yields

$\Delta T = \left(\frac{G^2S^2T_{Atm}}{A^2c^2\rho_{Hot}^2 2gh}\right)^{1/3} \approx 9.1$ K.	(20)

From this we get

$w = 760$ kg/s.	(21)

Solution - model of an atomic nucleus

Solution of Task 1

a) In the SC-system, in each of 8 corners of a given cube there is one unit (atom, nucleon, etc.), but it is shared by 8 neighboring cubes – this gives a total of one nucleon per cube. If nucleons are touching, as we assume in our simplified model, then $a = 2r_N$ is the cube edge length a. The volume of one nucleon is then

$V_N = \frac{4}{3} r_N^3 \pi = \frac{4}{3}\left(\frac{a}{2}\right)^3 \pi = \frac{4a^3}{3 \cdot 8}\pi = \frac{\pi}{6} a^3$		(1)

from which we obtain

$f = \frac{V_N}{a^3} = \frac{\pi}{6} \approx 0.52$		(2)

b) The mass density of the nucleus is:

$\rho_m = f\frac{m_N}{V_N} = 0.52 \cdot \frac{1.67 \cdot 10^{-27}}{4/3 \cdot \left(0.85 \cdot 10^{-15}\right)^3 \pi} \approx 3.40 \cdot 10^{17}\ \frac{\mathrm{kg}}{\mathrm{m}^3}$.		(4)

c) Taking into account the approximation that the number of protons and neutrons is approximately equal, for charge density we get:

$\rho_c = \frac{f}{2}\frac{e}{V_N} = \frac{0.52}{2} \cdot \frac{1.6 \cdot 10^{-19}}{4/3 \cdot \left(0.85 \cdot 10^{-15}\right)^3 \pi} \approx 1.63 \cdot 10^{25}\ \frac{\mathrm{C}}{\mathrm{m}^3}$		(5)

The number of nucleons in a given nucleus is *A*. The total volume occupied by the nucleus is:

$V = \frac{AV_N}{f}$,		(6)

which gives the following relation between radii of nucleus and the number of nucleons:

$R = r_N\left(\frac{A}{f}\right)^{1/3} = \frac{r_N}{f^{1/3}} A^{1/3} = \frac{0.85}{0.52^{1/3}} A^{1/3} = 1.06\,\mathrm{fm} \cdot A^{1/3}$.		(7)

The numerical constant (1.06 fm) in the equation above will be denoted as r_0 in the sequel.

Solution of Task 2

First one needs to estimate the number of surface nucleons. The surface nucleons are in a spherical shell of width $2r_N$ at the surface. The volume of this shell is

$V_{surface} = \frac{4}{3}R^3\pi - \frac{4}{3}(R-2r_N)^3\pi =$ $= \frac{4}{3}R^3\pi - \frac{4}{3}R^3\pi + \frac{4}{3}\pi 3R^2 2r_N - \frac{4}{3}\pi 3R4r_N^2 + \frac{4}{3}\pi 8r_N^3$ $= 8\pi R r_N(R-2r_N) + \frac{4}{3}\pi 8r_N^3 =$ $= 8\pi(R^2 r_N - 2Rr_N^2 + \frac{4}{3}r_N^3)$	(8)

The number of surface nucleons is:

$A_{surface} = f\frac{V_{surface}}{V_N} = f\frac{8\pi(R^2r_N - 2Rr_N^2 + \frac{4}{3}r_N^3)}{\frac{4}{3}r_N^3\pi} =$ $= f6\left(\left(\frac{R}{r_N}\right)^2 - 2\left(\frac{R}{r_N}\right) + \frac{4}{3}\right) =$ $= f6\left(\left(\frac{A}{f}\right)^{2/3} - 2\left(\frac{A}{f}\right)^{1/3} + \frac{4}{3}\right) =$ $= 6f^{1/3}A^{2/3} - 12f^{2/3}A^{1/3} + 8f =$ $= 6^{2/3}\pi^{1/3}A^{2/3} - 2\cdot 6^{1/3}\pi^{2/3}A^{1/3} + \frac{4}{3}\pi \approx$ $\approx 4.84A^{2/3} - 7.80A^{1/3} + 4.19.$	(9)

The binding energy is now:

$E_b = (A - A_{surface})a_V + A_{surface}\frac{a_V}{2} =$ $= Aa_V - A_{surface}\frac{a_V}{2} =$ $= Aa_V - (3f^{1/3}A^{2/3} - 6f^{2/3}A^{1/3} + 4f)a_V =$ $= Aa_V - 3f^{1/3}A^{2/3}a_V + 6f^{2/3}A^{1/3}a_V - 4fa_V =$ $= (15.8A - 38.20A^{2/3} + 61.58A^{1/3} - 33.09)\mathrm{MeV}$	(10)

Solution of Task 3 - Electrostatic (Coulomb) effects on the binding energy

a) Replacing Q_0 with Ze gives the electrostatic energy of the nucleus as:

$U_c = \frac{3(Ze)^2}{20\pi\varepsilon_0 R} = \frac{3Z^2e^2}{20\pi\varepsilon_0 R}$	(12)

The fact that each proton is not acting upon itself is taken into account by replacing Z^2 with $Z(Z-1)$:

$U_c = \dfrac{3Z(Z-1)e^2}{20\pi\varepsilon_0 R}$	(13)

b) In the formula for the electrostatic energy we should replace R with $r_N f^{-1/3} A^{1/3}$ to obtain

$\Delta E_b = -\dfrac{3e^2 f^{1/3}}{20\pi\varepsilon_0 r_N}\dfrac{Z(Z-1)}{A^{1/3}} = -\dfrac{Z(Z-1)}{A^{1/3}}\cdot 1.31\times 10^{-13}\text{ J}$ $= -\dfrac{Z(Z-1)}{A^{1/3}}\cdot 0.815\ \text{MeV} \approx \text{-}0.204\text{A}^{5/3}\text{MeV} + 0.409\text{A}^{2/3}\ \text{MeV}$	(14)

where *Z≈A/2* has been used. The Coulomb repulsion reduces the binding energy, hence the negative sign before the first (main) term. The complete formula for binding energy now gives:

$E_b = Aa_V - 3f^{1/3}A^{2/3}a_V + 6f^{2/3}A^{1/3}a_V - 4fa_V - \dfrac{3e^2 f^{1/3}}{20\pi\varepsilon_0 r_N}\left(\dfrac{A^{5/3}}{4} - \dfrac{A^{2/3}}{2}\right)$	(15)

Solution of Task 4 - Fission of heavy nuclei

a) The kinetic energy comes from the difference of binding energies (2 small nuclei – the original large one) and the Coulomb energy between two smaller nuclei (with *Z/2=A/4* nucleons each):

$E_{kin}(d) = 2E_b\left(\dfrac{A}{2}\right) - E_b(A) - \dfrac{1}{4\pi\varepsilon_0}\dfrac{A^2e^2}{4\cdot 4\cdot d} =$ $= -3f^{1/3}A^{2/3}a_V(2^{1/3}-1) + 6f^{2/3}A^{1/3}a_V(2^{2/3}-1)$ $-4fa_V - \dfrac{3e^2f^{1/3}}{20\pi\varepsilon_0 r_N}\left[\dfrac{A^{5/3}}{4}\left(2^{-2/3}-1\right) - \dfrac{A^{2/3}}{2}\left(2^{1/3}-1\right)\right]$ $-\dfrac{1}{4\pi\varepsilon_0}\dfrac{A^2e^2}{16d}$	(16)

(notice that the first term, Aa_V, cancels out).

b) The kinetic energy when $d = 2R(A/2)$ is given with:

$E_{kin} = 2E_b\left(\dfrac{A}{2}\right) - E_b(A) - \dfrac{1}{4\pi\varepsilon_0}\dfrac{2^{1/3}A^2e^2}{16\cdot 2r_N A^{1/3} f^{-1/3}} =$ $= -3f^{1/3}A^{2/3}a_V(2^{1/3}-1) + 6f^{2/3}A^{1/3}a_V(2^{2/3}-1)$ $-4fa_V - \dfrac{e^2f^{1/3}}{\pi\varepsilon_0 r_N}\left[\dfrac{3}{80}(2^{-2/3}-1) + \dfrac{2^{1/3}}{128}\right]A^{5/3} - \dfrac{e^2f^{1/3}}{\pi\varepsilon_0 r_N}\left[\dfrac{3}{40}(2^{1/3}-1)\right]A^{2/3} =$ $= (0.02203A^{5/3} - 10.0365A^{2/3} + 36.175A^{1/3} - 33.091)\text{MeV}$	(17)

Numerically one gets:

A=100 … E_{kin}= -33.95 MeV,

A=150 … E_{kin}= -30.93 MeV,

A=200 ... E_{kin}= -14.10 MeV,
A=250 ... E_{kin}= +15.06 MeV.

In our model, fission is possible when $E_{kin}(d = 2R(A/2)) \geq 0$. From the numerical evaluations given above, one sees that this happens approximately halfway between A=200 and A=250 – a rough estimate would be A≈225. Precise numerical evaluation of the equation:

$E_{kin} = (0.02203A^{5/3} - 10.0365A^{2/3} + 36.175A^{1/3} - 33.091)\mathrm{MeV} \geq 0$	(18)

gives that for $A \geq 227$ fission is possible.

Solution of Task 5 – Transfer reactions

Task 5a) This part can be solved by using either non-relativistic or relativistic kinematics.

Non-relativistic solution

First one has to find the amount of mass transferred to energy in the reaction (or the energy equivalent, so-called Q-value):

$\Delta m = (\text{total mass})_{\text{after reaction}} - (\text{total mass})_{\text{before reaction}} =$ $= (57.93535 + 12.00000)\ \text{a.m.u.} - (53.93962 + 15.99491)\ \text{a.m.u.} =$ $= 0.00082\ \text{a.m.u.} =$ $= 1.3616 \cdot 10^{-30}\ \text{kg}.$	(19)

Using the Einstein formula for equivalence of mass and energy, we get:

$Q = (\text{total kinetic energy})_{\text{after reaction}} - (\text{total kinetic energy})_{\text{before reaction}} =$ $= -\Delta m \cdot c^2 =$ $= -1.3616 \cdot 10^{-30} \cdot 299792458^2 = -1.2237 \cdot 10^{-13}\ \text{J}$	(20)

Taking into account that 1 MeV is equal to $1.602 \cdot 10^{-13}$ J, we get:

$Q = -1.2237 \cdot 10^{-13} / 1.602 \cdot 10^{-13} = -0.761\,\mathrm{MeV}$	(21)

This exercise is now solved using the laws of conservation of energy and momentum. The latter gives (we are interested only for the case when ^{12}C and ^{16}O are having the same direction so we don't need to use vectors):

$m(^{16}\mathrm{O})v(^{16}\mathrm{O}) = m(^{12}\mathrm{C})v(^{12}\mathrm{C}) + m(^{58}\mathrm{Ni})v(^{58}\mathrm{Ni})$	(22)

while the conservation of energy gives:

$E_k(^{16}\mathrm{O}) + Q = E_k(^{12}\mathrm{C}) + E_k(^{58}\mathrm{Ni}) + E_x(^{58}\mathrm{Ni})$	(23)

where $E_x(^{58}\mathrm{Ni})$ is the excitation energy of ^{58}Ni, and Q is calculated in the first part of this task. But since ^{12}C and ^{16}O have the same velocity, conservation of momentum reduced to:

$$\left[m\left(^{16}\mathrm{O}\right)-m\left(^{12}\mathrm{C}\right)\right]v\left(^{16}\mathrm{O}\right)=m\left(^{58}\mathrm{Ni}\right)v\left(^{58}\mathrm{Ni}\right) \tag{24}$$

Now we can easily find the kinetic energy of ^{58}Ni:

$$\begin{aligned}E_k\left(^{58}\mathrm{Ni}\right)&=\frac{m\left(^{58}\mathrm{Ni}\right)v^2\left(^{58}\mathrm{Ni}\right)}{2}=\frac{\left[m\left(^{58}\mathrm{Ni}\right)v\left(^{58}\mathrm{Ni}\right)\right]^2}{2m\left(^{58}\mathrm{Ni}\right)}=\\&=\frac{\left[\left[m\left(^{16}\mathrm{O}\right)-m\left(^{12}\mathrm{C}\right)\right]v\left(^{16}\mathrm{O}\right)\right]^2}{2m\left(^{58}\mathrm{Ni}\right)}=\\&=E_k\left(^{16}\mathrm{O}\right)\frac{\left[m\left(^{16}\mathrm{O}\right)-m\left(^{12}\mathrm{C}\right)\right]^2}{m\left(^{58}\mathrm{Ni}\right)m\left(^{16}\mathrm{O}\right)}\end{aligned} \tag{25}$$

and finally the excitation energy of ^{58}Ni:

$$\begin{aligned}E_x\left(^{58}\mathrm{Ni}\right)&=E_k\left(^{16}\mathrm{O}\right)+Q-E_k\left(^{12}\mathrm{C}\right)-E_k\left(^{58}\mathrm{Ni}\right)=\\&=E_k\left(^{16}\mathrm{O}\right)+Q-\frac{m\left(^{12}\mathrm{C}\right)v^2\left(^{16}\mathrm{O}\right)}{2}-E_k\left(^{16}\mathrm{O}\right)\frac{\left[m\left(^{16}\mathrm{O}\right)-m\left(^{12}\mathrm{C}\right)\right]^2}{m\left(^{58}\mathrm{Ni}\right)m\left(^{16}\mathrm{O}\right)}=\\&=Q+E_k\left(^{16}\mathrm{O}\right)-E_k\left(^{16}\mathrm{O}\right)\cdot\frac{m\left(^{12}\mathrm{C}\right)}{m\left(^{16}\mathrm{O}\right)}-E_k\left(^{16}\mathrm{O}\right)\frac{\left[m\left(^{16}\mathrm{O}\right)-m\left(^{12}\mathrm{C}\right)\right]^2}{m\left(^{58}\mathrm{Ni}\right)m\left(^{16}\mathrm{O}\right)}=\\&=Q+E_k\left(^{16}\mathrm{O}\right)\left[1-\frac{m\left(^{12}\mathrm{C}\right)}{m\left(^{16}\mathrm{O}\right)}-\frac{\left[m\left(^{16}\mathrm{O}\right)-m\left(^{12}\mathrm{C}\right)\right]^2}{m\left(^{58}\mathrm{Ni}\right)m\left(^{16}\mathrm{O}\right)}\right]=\\&=Q+E_k\left(^{16}\mathrm{O}\right)\frac{\left[m\left(^{16}\mathrm{O}\right)-m\left(^{12}\mathrm{C}\right)\right]\cdot\left[m\left(^{58}\mathrm{Ni}\right)-m\left(^{16}\mathrm{O}\right)+m\left(^{12}\mathrm{C}\right)\right]}{m\left(^{58}\mathrm{Ni}\right)m\left(^{16}\mathrm{O}\right)}\end{aligned} \tag{26}$$

Note that the first bracket in numerator is approximately equal to the mass of transferred particle (the ^{4}He nucleus), while the second one is approximately equal to the mass of target nucleus ^{54}Fe. Inserting the numbers we get:

$$\begin{aligned}E_x\left(^{58}\mathrm{Ni}\right)&=-0.761+50\cdot\frac{(15.99491-12.)(57.93535-15.99491+12.)}{57.93535\cdot 15.99491}=\\&=10.866\,\mathrm{MeV}\end{aligned} \tag{27}$$

Relativistic solution

In the relativistic version, solution is found starting from the following pair of equations (the first one is the law of conservation of energy and the second one the law of conservation of momentum):

$$m\left(^{54}\mathrm{Fe}\right)\cdot c^2+\frac{m\left(^{16}\mathrm{O}\right)\cdot c^2}{\sqrt{1-v^2\left(^{16}\mathrm{O}\right)/c^2}}=\frac{m\left(^{12}\mathrm{C}\right)\cdot c^2}{\sqrt{1-v^2\left(^{12}\mathrm{C}\right)/c^2}}+\frac{m^*\left(^{58}\mathrm{Ni}\right)\cdot c^2}{\sqrt{1-v^2\left(^{58}\mathrm{Ni}\right)/c^2}} \tag{28}$$

$$\frac{m(^{16}\mathrm{O})\cdot v(^{16}\mathrm{O})}{\sqrt{1-v^2(^{16}\mathrm{O})/c^2}} = \frac{m(^{12}\mathrm{C})\cdot v(^{12}\mathrm{C})}{\sqrt{1-v^2(^{12}\mathrm{C})/c^2}} + \frac{m^*(^{58}\mathrm{Ni})\cdot v(^{58}\mathrm{Ni})}{\sqrt{1-v^2(^{58}\mathrm{Ni})/c^2}}$$

All the masses in the equations are the rest masses; the ^{58}Ni is NOT in its ground-state, but in one of its excited states (having the mass denoted with *m**). Since ^{12}C and ^{16}O have the same velocity, this set of equations reduces to:

$$m(^{54}\mathrm{Fe}) + \frac{m(^{16}\mathrm{O}) - m(^{12}\mathrm{C})}{\sqrt{1-v^2(^{16}\mathrm{O})/c^2}} = \frac{m^*(^{58}\mathrm{Ni})}{\sqrt{1-v^2(^{58}\mathrm{Ni})/c^2}}$$
$$\frac{\left(m(^{16}\mathrm{O}) - m(^{12}\mathrm{C})\right)\cdot v(^{16}\mathrm{O})}{\sqrt{1-v^2(^{16}\mathrm{O})/c^2}} = \frac{m^*(^{58}\mathrm{Ni})\cdot v(^{58}\mathrm{Ni})}{\sqrt{1-v^2(^{58}\mathrm{Ni})/c^2}} \qquad (29)$$

Dividing the second equation with the first one gives:

$$v(^{58}\mathrm{Ni}) = \frac{\left(m(^{16}\mathrm{O}) - m(^{12}\mathrm{C})\right)\cdot v(^{16}\mathrm{O})}{\left(m(^{16}\mathrm{O}) - m(^{12}\mathrm{C})\right) + m(^{54}\mathrm{Fe})\sqrt{1-v^2(^{16}\mathrm{O})/c^2}} \qquad (30)$$

The velocity of projectile can be calculated from its energy:

$$E_{kin}(^{16}\mathrm{O}) = \frac{m(^{16}\mathrm{O})\cdot c^2}{\sqrt{1-v^2(^{16}\mathrm{O})/c^2}} - m(^{16}\mathrm{O})\cdot c^2$$
$$\sqrt{1-v^2(^{16}\mathrm{O})/c^2} = \frac{m(^{16}\mathrm{O})\cdot c^2}{E_{kin}(^{16}\mathrm{O}) + m(^{16}\mathrm{O})\cdot c^2}$$
$$v^2(^{16}\mathrm{O})/c^2 = 1 - \left(\frac{m(^{16}\mathrm{O})\cdot c^2}{E_{kin}(^{16}\mathrm{O}) + m(^{16}\mathrm{O})\cdot c^2}\right)^2$$
$$v(^{16}\mathrm{O}) = \sqrt{1 - \left(\frac{m(^{16}\mathrm{O})\cdot c^2}{E_{kin}(^{16}\mathrm{O}) + m(^{16}\mathrm{O})\cdot c^2}\right)^2}\cdot c \qquad (31)$$

For the given numbers we get:

$$v(^{16}\mathrm{O}) = \sqrt{1 - \left(\frac{15.99491\cdot 1.6605\cdot 10^{-27}\cdot (2.9979\cdot 10^8)^2}{50\cdot 1.602\cdot 10^{-13} + 15.99491\cdot (2.9979\cdot 10^8)^2}\right)^2}\cdot c =$$
$$= \sqrt{1-0.99666^2}\cdot c = 0.08172\cdot c = 2.4498\cdot 10^7\ \mathrm{km/s} \qquad (32)$$

Now we can calculate:

$$v(^{58}\mathrm{Ni}) = \frac{(15.99491-12.0)\cdot 2.4498\cdot 10^7\ \mathrm{km/s}}{(15.99491-12.0) + 53.93962\sqrt{1-0.08172^2}} = 1.6946\cdot 10^6\ \mathrm{km/s} \qquad (33)$$

The mass of ^{58}Ni in its excited state is then:

$$m^*\left({}^{58}\mathrm{Ni}\right) = \left(m\left({}^{16}\mathrm{O}\right) - m\left({}^{12}\mathrm{C}\right)\right)\frac{\sqrt{1 - v^2\left({}^{58}\mathrm{Ni}\right)/c^2}}{\sqrt{1 - v^2\left({}^{16}\mathrm{O}\right)/c^2}} \cdot \frac{v\left({}^{16}\mathrm{O}\right)}{v\left({}^{58}\mathrm{Ni}\right)} =$$
$$= (15.99491 - 12.0)\frac{\sqrt{1 - \left(1.6945\cdot 10^6 / 2.9979\cdot 10^8\right)^2}}{\sqrt{1 - 0.08172^2}} \cdot \frac{2.4498\cdot 10^7}{1.6945\cdot 10^6}\ \text{a.m.u.} =$$
$$= 57.9470\ \text{a.m.u.} \tag{34}$$

The excitation energy of ^{58}Ni is then:

$$E_x = \left[m^*\left({}^{58}\mathrm{Ni}\right) - m\left({}^{58}\mathrm{Ni}\right)\right]\cdot c^2 = (57.9470 - 57.93535)\cdot 1.6605\cdot 10^{-27}\left(2.9979\cdot 10^8\right)^2 =$$
$$= 2.00722\cdot 10^{-12}/1.602\cdot 10^{-13}\ \mathrm{MeV/J} = 10.8636\ \mathrm{MeV} \tag{35}$$

The relativistic and non-relativistic results are equal within 2 keV so both can be considered as correct –we can conclude that at the given beam energy, relativistic effects are not important.

Task 5b) For gamma-emission from the static nucleus, laws of conservation of energy and momentum give:

$$E_x\left({}^{58}\mathrm{Ni}\right) = E_\gamma + E_{\mathrm{recoil}}$$
$$p_\gamma = p_{\mathrm{recoil}} \tag{36}$$

Gamma-ray and recoiled nucleus have, of course, opposite directions. For gamma-ray (photon), energy and momentum are related as:

$$E_\gamma = p_\gamma \cdot c \tag{37}$$

In part a) we have seen that the nucleus motion in this energy range is not relativistic, so we have:

$$E_{\mathrm{recoil}} = \frac{p_{\mathrm{recoil}}^2}{2m\left({}^{58}\mathrm{Ni}\right)} = \frac{p_\gamma^2}{2m\left({}^{58}\mathrm{Ni}\right)} = \frac{E_\gamma^2}{2m\left({}^{58}\mathrm{Ni}\right)\cdot c^2} \tag{38}$$

Inserting this into law of energy conservation Eq. (36), we get:

$$E_x\left({}^{58}\mathrm{Ni}\right) = E_\gamma + E_{\mathrm{recoil}} = E_\gamma + \frac{E_\gamma^2}{2m\left({}^{58}\mathrm{Ni}\right)\cdot c^2} \tag{39}$$

This reduces to the quadratic equation:

$$E_\gamma^2 + 2m\left({}^{58}\mathrm{Ni}\right)c^2 \cdot E_\gamma + 2m\left({}^{58}\mathrm{Ni}\right)c^2 E_x\left({}^{58}\mathrm{Ni}\right) = 0 \qquad (40)$$

which gives the following solution:

$$E_\gamma = \frac{-2m\left({}^{58}\mathrm{Ni}\right)c^2 + \sqrt{4\left(m\left({}^{58}\mathrm{Ni}\right)c^2\right)^2 + 8m\left({}^{58}\mathrm{Ni}\right)c^2 E_x\left({}^{58}\mathrm{Ni}\right)}}{2} = \\ = \sqrt{\left(m\left({}^{58}\mathrm{Ni}\right)c^2\right)^2 + 2m\left({}^{58}\mathrm{Ni}\right)c^2 E_x\left({}^{58}\mathrm{Ni}\right)} - m\left({}^{58}\mathrm{Ni}\right)c^2 \qquad (41)$$

Inserting numbers gives:

$$E_\gamma = 10.8633\,\mathrm{MeV} \qquad (42)$$

The equation (37) can also be reduced to an approximate equation before inserting numbers:

$$E_\gamma = E_x\left(1 - \frac{E_x}{2m\left({}^{58}\mathrm{Ni}\right)c^2}\right) = 10.8633\,\mathrm{MeV} \qquad (43)$$

The recoil energy is now easily found as:

$$E_{\mathrm{recoil}} = E_x\left({}^{58}\mathrm{Ni}\right) - E_\gamma = 1.1\,\mathrm{keV} \qquad (44)$$

Due to the fact that nucleus emitting gamma-ray (^{58}Ni) is moving with the high velocity, the energy of gamma ray will be changed because of the Doppler effect. The relativistic Doppler effect (when source is moving towards observer/detector) is given with this formula:

$$f_{\mathrm{detector}} = f_{\gamma,\mathrm{emitted}}\sqrt{\frac{1+\beta}{1-\beta}} \qquad (45)$$

and since there is a simple relation between photon energy and frequency *(E=hf)*, we get the similar expression for energy:

$$E_{\mathrm{detector}} = E_{\gamma,\mathrm{emitted}}\sqrt{\frac{1+\beta}{1-\beta}} \qquad (46)$$

where $\beta = v/c$ and v is the velocity of emitter (the ^{58}Ni nucleus). Taking the calculated value of the ^{58}Ni velocity (equation 29) we get:

$$E_{\mathrm{detector}} = E_{\gamma,\mathrm{emitted}}\sqrt{\frac{1+\beta}{1-\beta}} = 10.863\sqrt{\frac{1+0.00565}{1-0.00565}} = 10.925\,\mathrm{MeV} \qquad (47)$$

Solution: Exp. problem 1

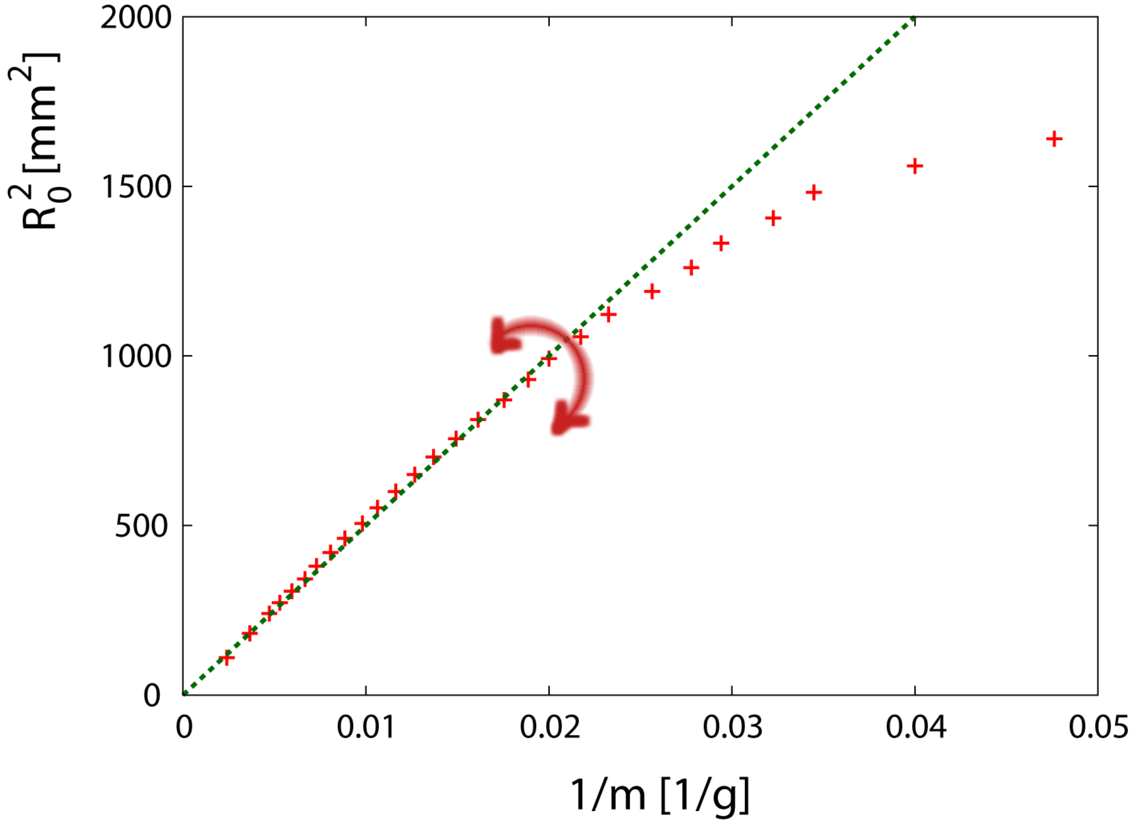
Task 1 Points

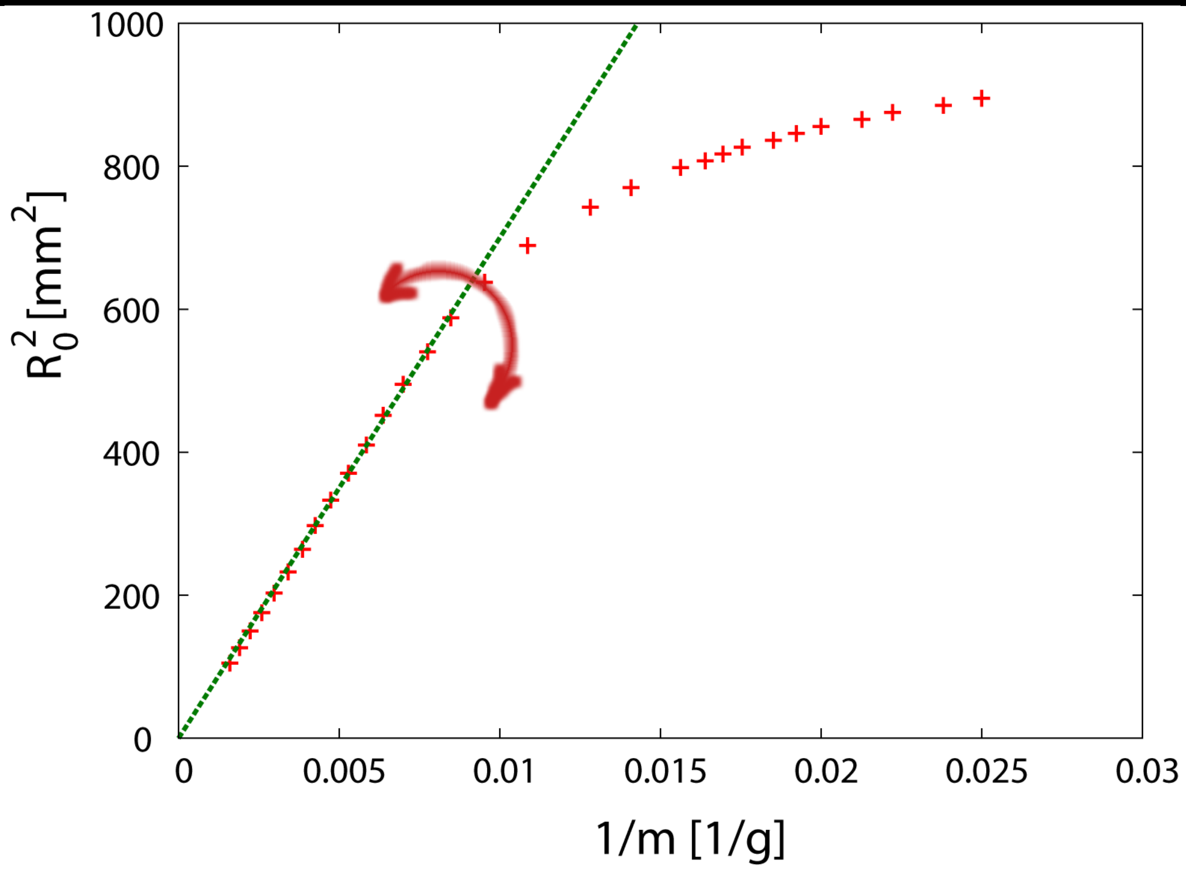
(a) 0.95

m[g]	R_0[mm]
21	40.5
25	39.5
29	38.5
31	37.5
34	36.5
36	35.5
39	34.5
43	33.5
46	32.5
50	31.5
53	30.5
57	29.5
62	28.5
67	27.5
73	26.5
79	25.5
86	24.5
94	23.5
102	22.5
113	21.5
124	20.5
137	19.5
150	18.5
168	17.5
189	16.5
212	15.5
274	13.5
417	10.5

Task 1		Points
(a)		0.95

m[g]	**R_0[mm]**
40	29.9
42	29.8
45	29.6
47	29.4
50	29.3
52	29.1
54	28.9
57	28.8
59	28.6
61	28.4
64	28.3
71	27.8
78	27.3
92	26.3
105	25.3
118	24.3
129	23.3
143	22.3
157	21.3
171	20.3
189	19.3
211	18.3
235	17.3
259	16.3
293	15.3
336	14.3
386	13.3
449	12.3

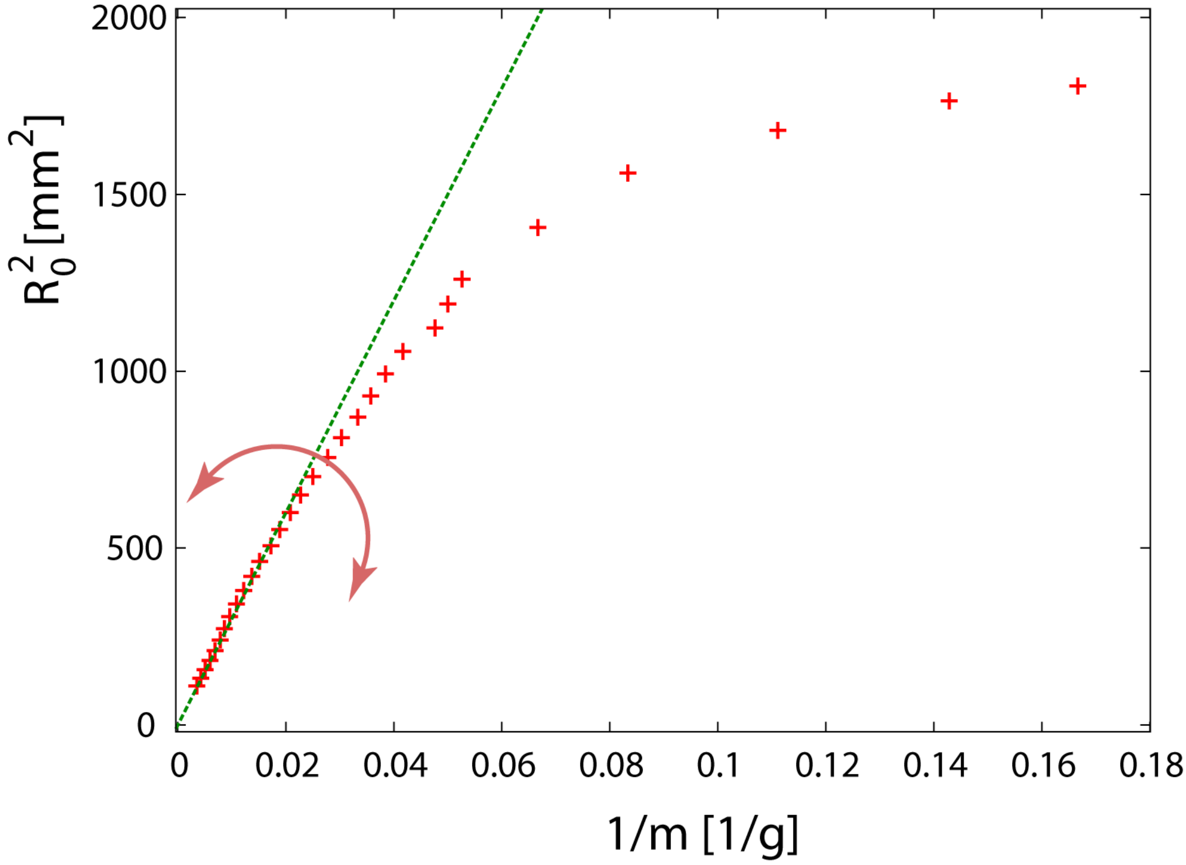
Task 1		Points
(b)	 $a = 50000$ g mm^2	1.4

Task 1		Points
(b)	 $a = 70000\ \text{g mm}^2$	1.4

Task 1		Points
(b)	$\kappa = \dfrac{2ag}{\pi d} =$ 1.5 mJ	0.5
	$\kappa = \dfrac{2ag}{\pi d} =$ 1.5 mJ	0.5
	$\dfrac{R_0}{R_c} \leq$ 0.70 $\dfrac{R_0}{R_c} \leq$ 0.77	0.5

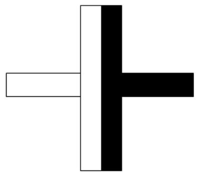
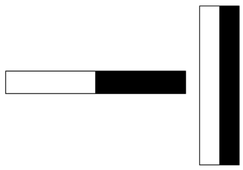
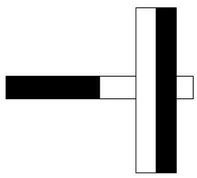
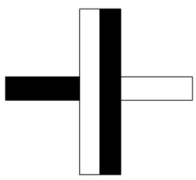
Task 2	Points
	0.9

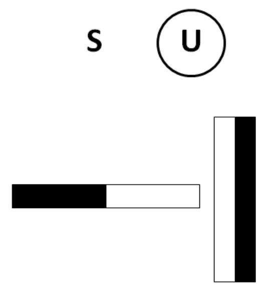
m[g]	R_0[mm]
6	42.5
7	42.
9	41.
12	39.5
15	37.5
19	35.5
20	34.5
21	33.5
24	32.5
26	31.5
28	30.5
30	29.5
33	28.5
36	27.5
40	26.5
44	25.5
48	24.5
53	23.5
58	22.5
66	21.5
73	20.5
82	19.5
92	18.5
104	17.5
116	16.5
127	15.5
145	14.5
168	13.5

Task 2		Points
	 $a = 27000\ \mathrm{g\ mm^2}$	0.9

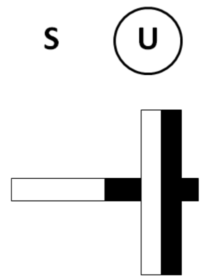
Task 2		Points
	$\kappa =$0.8 mJ	1.0
Task 3		**Points**
	Young modulus of the blue foil: Y=2.0 GPa Y=2.0 GPa	0.6
	Young modulus of the colorless foil: Y=2.5 GPa	0.4
Total:		10

Solution - Exp. Problem 2

Task 1		Points
	S U 	0.25
	S U 	0.45
	S U 	0.45
	S U 	0.45



0.45



0.45

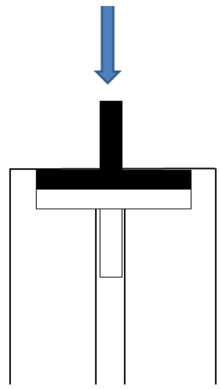
Task 2 **Points**

Symmetries that should be utilized in the measurements: 0,6

$F_{\uparrow\downarrow}(z) = -F_{\uparrow\downarrow}(-z)$
$F_{\uparrow\downarrow}(z) = -F_{\uparrow\uparrow}(z)$
From the two above one gets also $F_{\uparrow\uparrow}(z) = -F_{\uparrow\uparrow}(-z)$

By using the setup as it is, the whole curve can be measured by starting the measurements from three stable equilibrium points; the equilibrium point (z0) can be measured also by using the setup. 0,8

Configuration:



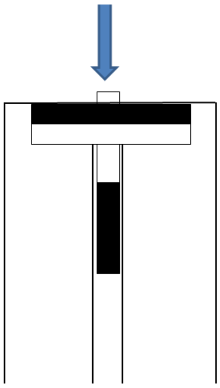
Measurements:

z0=0mm	m [g]	Δz [mm]
	0	0
	31	1
	55	2
	75	3
	97	4

119	5
140	6
158	7
171	8
170	9
118	10
85	10,25
50	10,5
10	10,75

0,8

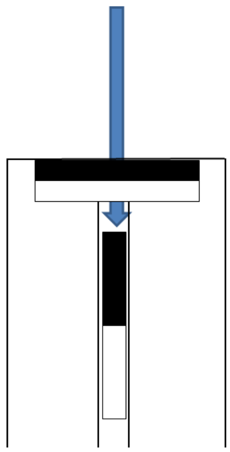
Configuration:



Measurements:

z0=10.8mm	m [g]	Δz [mm]
	0	0
	233	1
	538	2
	927	3
	996	3,5
	1124	4
	1154	4,5
	1213	5
	1212	5,5
	1120	6
	873	6,5
	284	7
	36	7,5

Configuration:

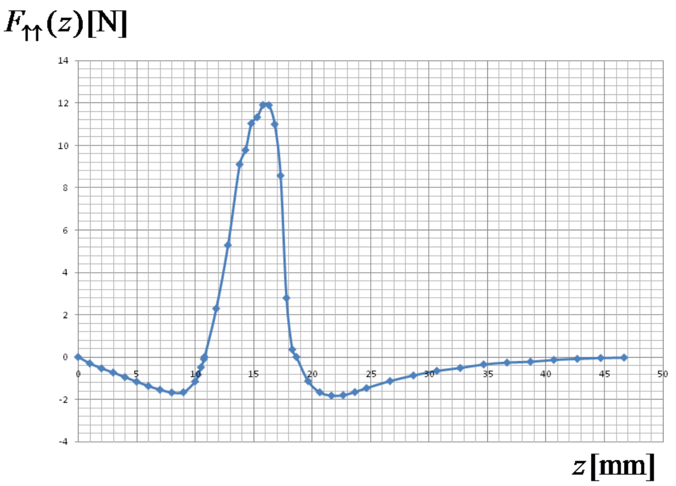
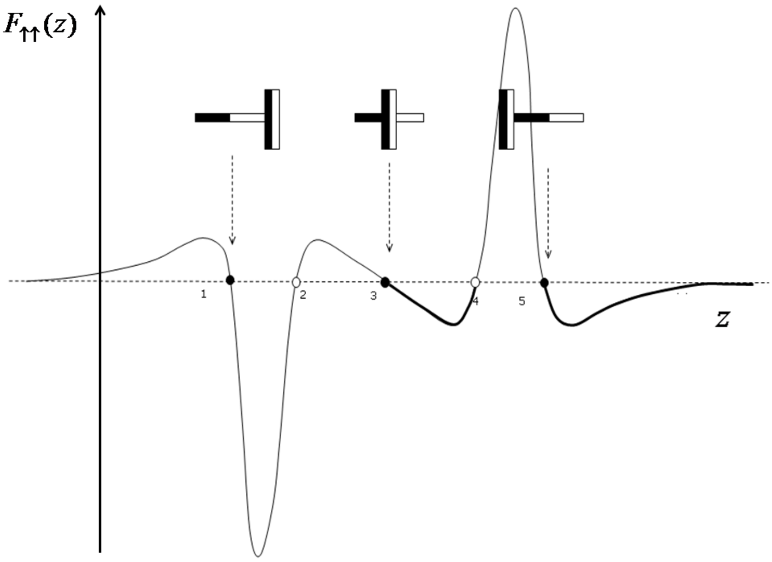
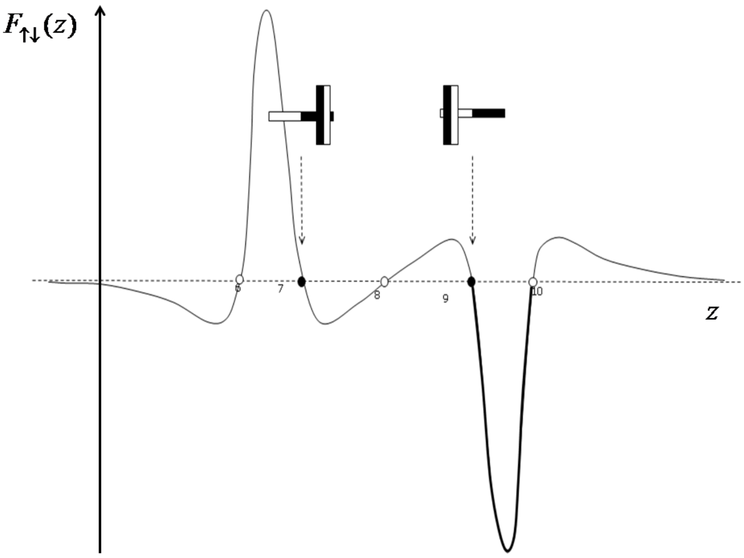


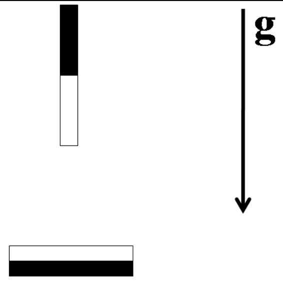
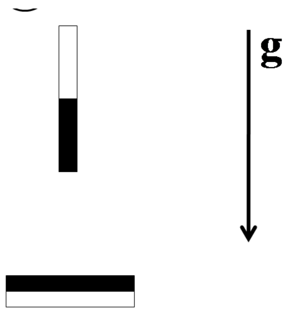
0,8

Measurements:

z0=18.6mm	m [g]	Δz [mm]
	0	0
	116	1
	170	2
	186	3
	184	4
	169	5
	150	6
	116	8
	89	10
	67	12
	53	14
	36	16
	27	18
	23	20
	14	22
	9	24
	5	26
	3	28

Task 3		Points
	Due to symmetry, it is sufficient to plot e.g., the following graph in detail:	2

		
		1
		1
Task 4		Points

	 	0,5
	OR  	
Total:		10.0

1. A Three-body Problem and LISA

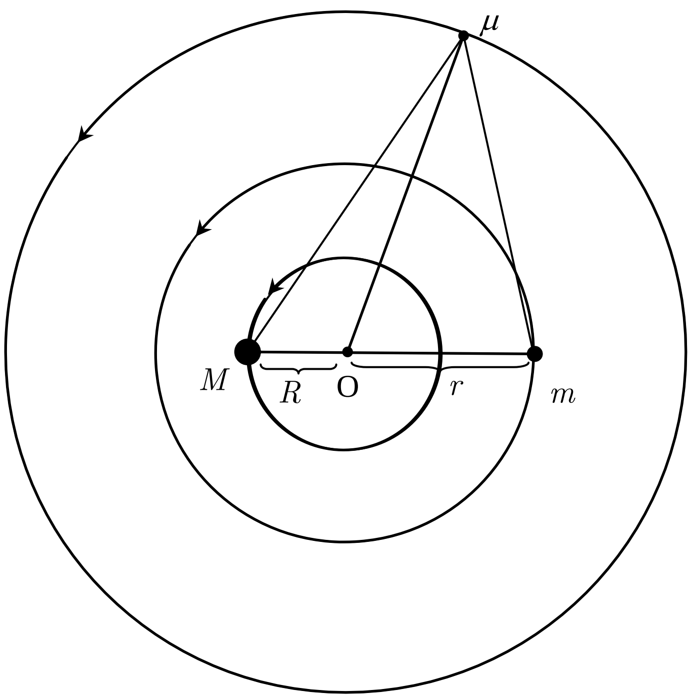


FIGURE 1 Coplanar orbits of three bodies.

1.1 Two gravitating masses M and m are moving in circular orbits of radii R and r, respectively, about their common centre of mass. Find the angular velocity ω_0 of the line joining M and m in terms of R, r, M, m and the universal gravitational constant G.

[1.5 points]

1.2 A third body of infinitesimal mass μ is placed in a coplanar circular orbit about the same centre of mass so that μ remains stationary relative to both M and m as shown in Figure 1. Assume that the infinitesimal mass is not collinear with M and m. Find the values of the following parameters in terms of R and r:

[3.5 points]

1.2.1 distance from μ to M.
1.2.2 distance from μ to m.
1.2.3 distance from μ to the centre of mass.

1.3 Consider the case $M = m$. If μ is now given a small radial perturbation (along Oμ), what is the angular frequency of oscillation of μ about the unperturbed position in terms of ω_0? Assume that the angular momentum of μ is conserved. **[3.2 points]**

The Laser Interferometry Space Antenna (LISA) is a group of three identical spacecrafts for detecting low frequency gravitational waves. Each of the spacecrafts is placed at the corners of an equilateral triangle as shown in Figure 2 and Figure 3. The sides (or 'arms') are about 5.0 million kilometres long. The LISA constellation is in an earth-like orbit around the Sun trailing the Earth by 20°. Each of them moves on a slightly inclined individual orbit around the Sun. Effectively, the three spacecrafts appear to roll about their common centre one revolution per year.

They are continuously transmitting and receiving laser signals between each other. Overall, they detect the gravitational waves by measuring tiny changes in the arm lengths using interferometric means. A collision of massive objects, such as blackholes, in nearby galaxies is an example of the sources of gravitational waves.

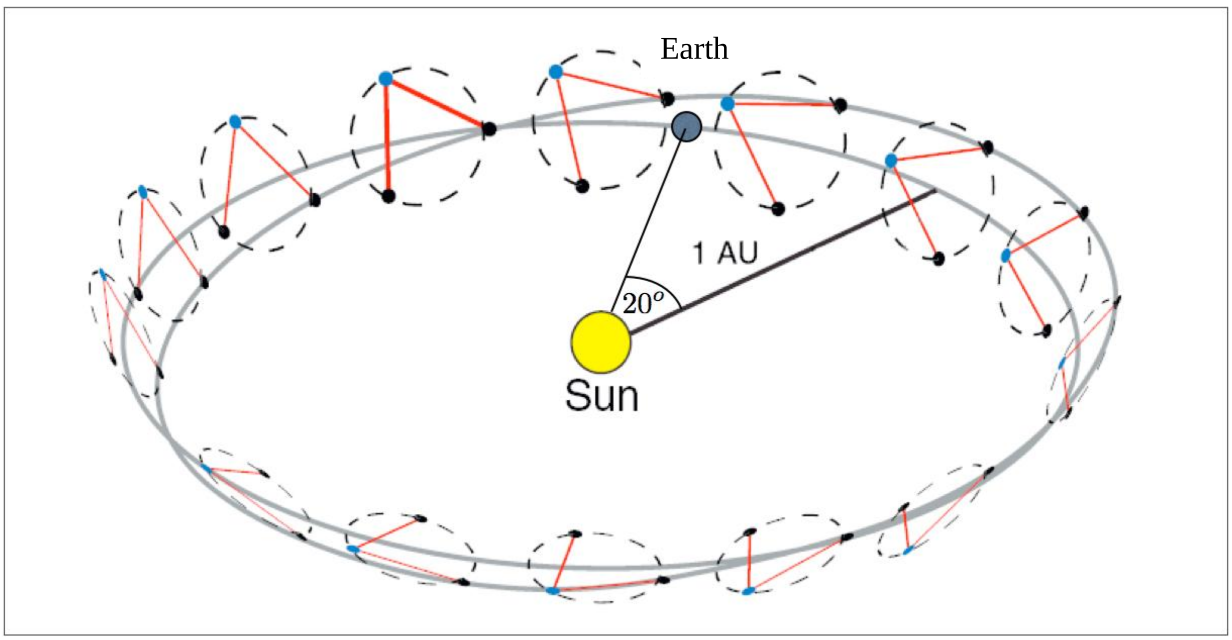


FIGURE 2 Illustration of the LISA orbit. The three spacecraft roll about their centre of mass with a period of 1 year. Initially, they trail the Earth by 20°. (Picture from D.A. Shaddock, "An Overview of the Laser Interferometer Space Antenna", *Publications of the Astronomical Society of Australia*, 2009, **26**, pp.128-132.).

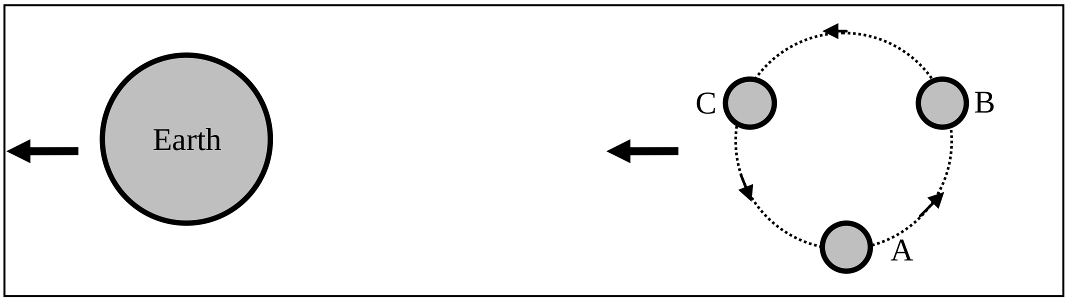
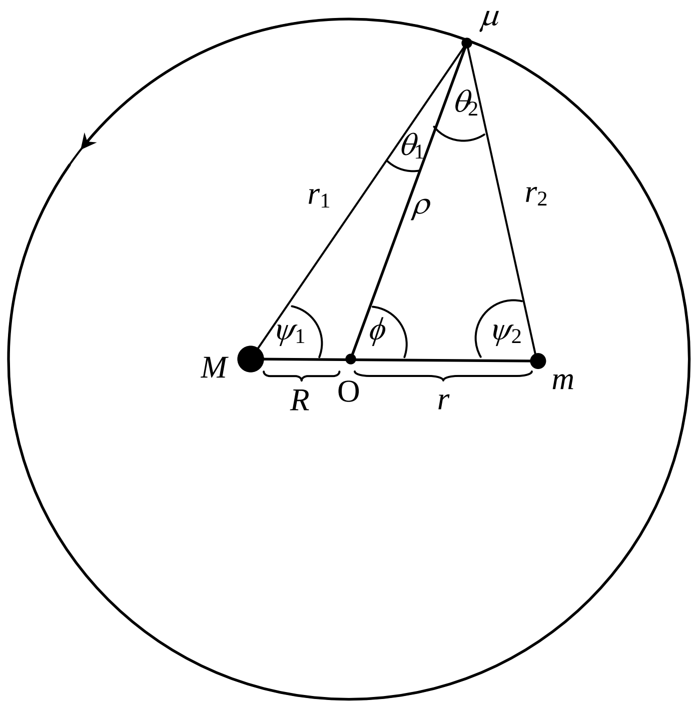


FIGURE 3 Enlarged view of the three spacecrafts trailing the Earth. A, B and C are the three spacecrafts at the corners of the equilateral triangle.

1.4 In the plane containing the three spacecrafts, what is the relative speed of one spacecraft with respect to another? **[1.8 point]**

I. Solution



1.1 Let O be their centre of mass. Hence

$$MR - mr = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$m\omega_0^2 r = \frac{GMm}{(R+r)^2}$$

$$M\omega_0^2 R = \frac{GMm}{(R+r)^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

From Eq. (2), or using reduced mass, $\omega_0^2 = \dfrac{G(M+m)}{(R+r)^3}$

Hence, $\omega_0^2 = \dfrac{G(M+m)}{(R+r)^3} = \dfrac{GM}{r(R+r)^2} = \dfrac{Gm}{R(R+r)^2}$. ………………………………… (3)

1.2 Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m. For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega_0^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \text{(4)}$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \text{(5)}$$

Substituting $\frac{GM}{r_1^2}$ from Eq. (5) into Eq. (4), and using the identity

$\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \sin(\theta_1+\theta_2)$, we get

$$m\frac{\sin(\theta_1+\theta_2)}{r_2^2} = \frac{(M+m)}{(R+r)^3}\rho\sin\theta_1 \qquad \text{(6)}$$

The distances r_2 and ρ, the angles θ_1 and θ_2 are related by two Sine Rule equations

$$\frac{\sin\psi_1}{\rho} = \frac{\sin\theta_1}{R}$$
$$\frac{\sin\psi_1}{r_2} = \frac{\sin(\theta_1+\theta_2)}{R+r} \qquad \text{(7)}$$

Substitute (7) into (6)

$$\frac{1}{r_2^3} = \frac{R}{(R+r)^4}\frac{(M+m)}{m} \qquad \text{(10)}$$

Since $\frac{m}{M+m} = \frac{R}{R+r}$, Eq. (10) gives

$$r_2 = R+r \qquad \text{(11)}$$

By substituting $\frac{Gm}{r_2^2}$ from Eq. (5) into Eq. (4), and repeat a similar procedure, we get

$$r_1 = R+r \qquad \text{(12)}$$

Alternatively, $\frac{r_1}{\sin(180^\circ-\phi)} = \frac{R}{\sin\theta_1}$ and $\frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2}$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1}$$

Combining with Eq. (5) gives $r_1 = r_2$

Hence, it is an equilateral triangle with

$$\psi_1 = 60^\circ$$
$$\psi_2 = 60^\circ \qquad \text{.........................} \quad (13)$$

The distance ρ is calculated from the Cosine Rule.

$$\rho^2 = r^2 + (R+r)^2 - 2r(R+r)\cos 60^\circ$$
$$\rho = \sqrt{r^2 + rR + R^2} \qquad \text{.........................} \quad (14)$$

Alternative Solution to 1.2

Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m .For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \text{.........................} \quad (4)$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \text{.........................} \quad (5)$$

Note that $$\frac{r_1}{\sin(180^\circ - \phi)} = \frac{R}{\sin\theta_1}$$

$$\frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2} \qquad \text{(see figure)}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1} \qquad \text{.........................} \quad (6)$$

Equations (5) and (6):

$$r_1 = r_2 \qquad \text{.........................} \quad (7)$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{m}{M} \qquad \text{.........................} \quad (8)$$

$$\psi_1 = \psi_2 \qquad \text{.........................} \quad (9)$$

The equation (4) then becomes:

$$M\cos\theta_1 + m\cos\theta_2 = \frac{(M+m)}{(R+r)^3}r_1^2\rho \qquad \text{.........................} \quad (10)$$

Equations (8) and (10): $$\sin(\theta_1+\theta_2) = \frac{M+m}{M}\frac{r_1^2\rho}{(R+r)^3}\sin\theta_2 \qquad \text{.........................} \quad (11)$$

Note that from figure, $$\frac{\rho}{\sin\psi_2} = \frac{r}{\sin\theta_2} \qquad \text{.........................} \quad (12)$$

Equations (11) and (12): $\sin(\theta_1+\theta_2) = \dfrac{M+m}{M}\dfrac{r_1^2 r}{(R+r)^3}\sin\psi_2$ ……………………… (13)

Also from figure,

$$(R+r)^2 = r_2^2 - 2r_1r_2\cos(\theta_1+\theta_2) + r_1^2 = 2r_1^2\left[1-\cos(\theta_1+\theta_2)\right] \quad \text{……………… (14)}$$

Equations (13) and (14): $\sin(\theta_1+\theta_2) = \dfrac{\sin\psi_2}{2\left[1-\cos(\theta_1+\theta_2)\right]}$ ……………………… (15)

$$\theta_1+\theta_2 = 180^\circ - \psi_1 - \psi_2 = 180^\circ - 2\psi_2 \quad \text{(see figure)}$$

$$\therefore \quad \cos\psi_2 = \frac{1}{2},\ \psi_2 = 60^\circ,\ \psi_1 = 60^\circ$$

Hence M and m from an equilateral triangle of sides $(R+r)$

Distance μ to M is $R+r$

Distance μ to m is $R+r$

Distance μ to O is $\rho = \sqrt{\left(\dfrac{R+r}{2}-R\right)^2 + \left\{(R+r)\dfrac{\sqrt{3}}{2}\right\}^2} = \sqrt{R^2+Rr+r^2}$

1.3 The energy of the mass μ is given by

$$E = -\frac{GM\mu}{r_1} - \frac{Gm\mu}{r_2} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2 + \rho^2\omega^2\right) \quad \text{………………………..(15)}$$

Since the perturbation is in the radial direction, angular momentum is conserved ($r_1 = r_2 = \Re$ and $m = M$),

$$E = -\frac{2GM\mu}{\Re} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2 + \frac{\rho_0^4\omega_0^2}{\rho^2}\right) \quad \text{………………………..(16)}$$

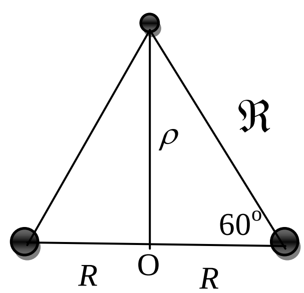
Since the energy is conserved,

$$\frac{dE}{dt} = 0$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\Re^2}\frac{d\Re}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0^4\omega_0^2}{\rho^3}\frac{d\rho}{dt} = 0 \quad \text{……………(17)}$$

$$\frac{d\Re}{dt} = \frac{d\Re}{d\rho}\frac{d\rho}{dt} = \frac{d\rho}{dt}\frac{\rho}{\Re} \quad \text{……………(18)}$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\Re^3}\rho\frac{d\rho}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0^4\omega_0^2}{\rho^3}\frac{d\rho}{dt} = 0 \quad \text{……………(19)}$$



Since $\dfrac{d\rho}{dt} \neq 0$, we have

$$\frac{2GM}{\Re^3}\rho + \frac{d^2\rho}{dt^2} - \frac{{\rho_0}^4{\omega_0}^2}{\rho^3} = 0 \text{ or}$$

$$\frac{d^2\rho}{dt^2} = -\frac{2GM}{\Re^3}\rho + \frac{{\rho_0}^4{\omega_0}^2}{\rho^3}. \qquad \text{......(20)}$$

The perturbation from $\Re_0$ and ρ_0 gives $\Re = \Re_0\left(1+\dfrac{\Delta\Re}{\Re_0}\right)$ and $\rho = \rho_0\left(1+\dfrac{\Delta\rho}{\rho_0}\right)$.

Then

$$\frac{d^2\rho}{dt^2} = \frac{d^2}{dt^2}(\rho_0 + \Delta\rho) = -\frac{2GM}{\Re_0^3\left(1+\dfrac{\Delta\Re}{\Re_0}\right)^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right) + \frac{{\rho_0}^4{\omega_0}^2}{\rho_0^3\left(1+\dfrac{\Delta\rho}{\rho_0}\right)^3} \qquad \text{......(21)}$$

Using binomial expansion $(1+\varepsilon)^n \approx 1+n\varepsilon$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right)\left(1-\frac{3\Delta\Re}{\Re_0}\right) + \rho_0\omega_0^2\left(1-\frac{3\Delta\rho}{\rho_0}\right). \qquad \text{......(22)}$$

Using $\Delta\rho = \dfrac{\Re}{\rho}\Delta\Re$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \rho_0\omega_0^2\left(1-\frac{3\Delta\rho}{\rho_0}\right). \qquad \text{......(23)}$$

Since $\omega_0^2 = \dfrac{2GM}{\Re_0^3}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(1+\frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \omega_0^2\rho_0\left(1-\frac{3\Delta\rho}{\rho_0}\right) \qquad \text{......(24)}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(\frac{4\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) \qquad \text{......(25)}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4 - \frac{3\rho_0^2}{\Re_0^2}\right) \qquad \text{......(26)}$$

From the figure, $\rho_0 = \Re_0\cos 30°$ or $\dfrac{{\rho_0}^2}{{\Re_0}^2} = \dfrac{3}{4}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4-\frac{9}{4}\right) = -\frac{7}{4}\omega_0^2\Delta\rho. \qquad \text{......(27)}$$

Angular frequency of oscillation is $\frac{\sqrt{7}}{2}\omega_0$.

Alternative solution:

$M = m$ gives $R = r$ and $\omega_0^2 = \frac{G(M+M)}{(R+R)^3} = \frac{GM}{4R^3}$. The unperturbed radial distance of μ is $\sqrt{3}R$, so the perturbed radial distance can be represented by $\sqrt{3}R+\zeta$ where $\zeta << \sqrt{3}R$ as shown in the following figure.

Using Newton's 2nd law, $-\frac{2GM\mu}{\{R^2+(\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \mu\frac{d^2}{dt^2}(\sqrt{3}R+\zeta) - \mu\omega^2(\sqrt{3}R+\zeta)$.

(1)

The conservation of angular momentum gives $\mu\omega_0(\sqrt{3}R)^2 = \mu\omega(\sqrt{3}R+\zeta)^2$.

(2)

Manipulate (1) and (2) algebraically, applying $\zeta^2 \approx 0$ and binomial approximation.

$$-\frac{2GM}{\{R^2+(\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{2GM}{\{4R^2+2\sqrt{3}\zeta R\}^{3/2}}(\sqrt{3}R+\zeta) \approx \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{GM}{4R^3}\sqrt{3}R\frac{(1+\zeta/\sqrt{3}R)}{(1+\sqrt{3}\zeta/2R)^{3/2}} = \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\omega_0^2\sqrt{3}R\left(1-\frac{3\sqrt{3}\zeta}{4R}\right)\left(1+\frac{\zeta}{\sqrt{3}R}\right) \approx \frac{d^2\zeta}{dt^2} - \omega_0^2\sqrt{3}R\left(1-\frac{3\zeta}{\sqrt{3}R}\right)$$

$$\frac{d^2}{dt^2}\zeta = -\left(\frac{7}{4}\omega_0^2\right)\zeta$$

1.4 Relative velocity

Let v = speed of each spacecraft as it moves in circle around the centre O.
The relative velocities are denoted by the subscripts A, B and C.
For example, v_{BA} is the velocity of B as observed by A.

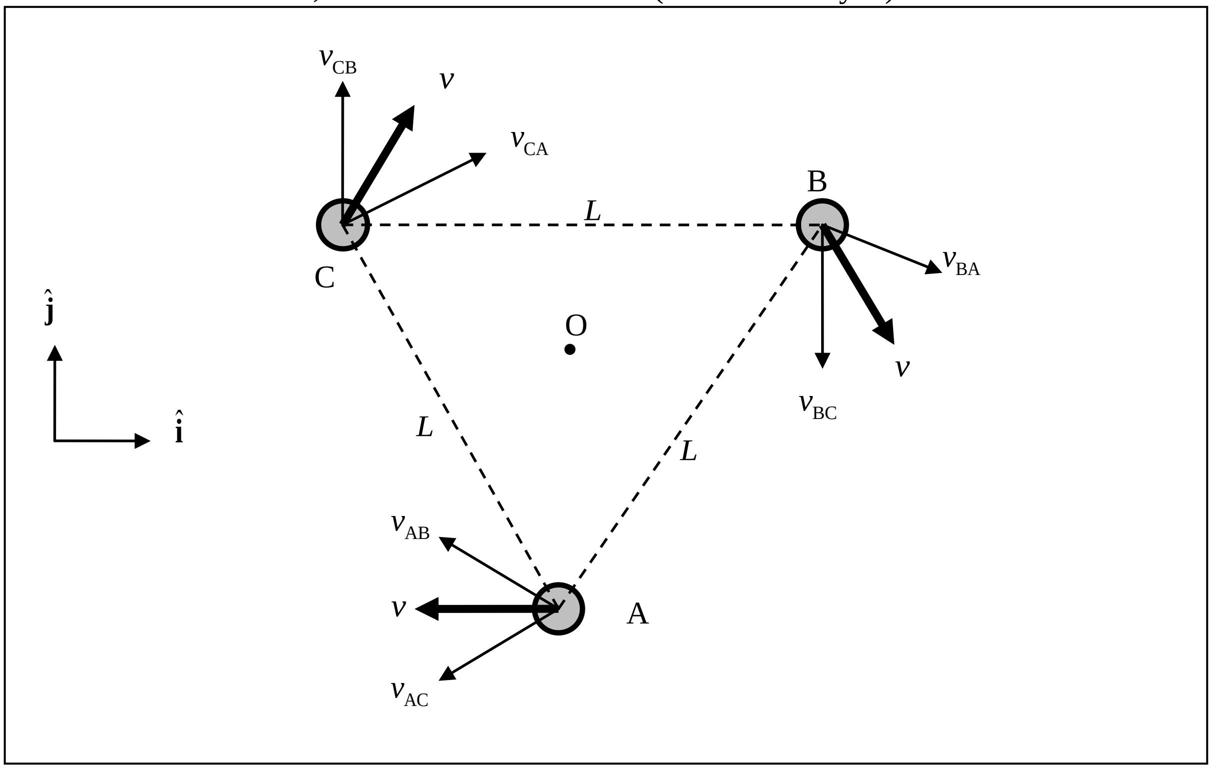
The period of circular motion is 1 year $T = 365\times24\times60\times60$ s. ………… (28)

The angular frequency $\omega = \frac{2\pi}{T}$

The speed $v = \omega\frac{L}{2\cos 30°} = 575$ m/s ………… (29)

The speed is much less than the speed light → Galilean transformation.

In Cartesian coordinates, the velocities of B and C (as observed by O) are



For B, $\vec{v}_B = v\cos 60°\hat{\mathbf{i}} - v\sin 60°\hat{\mathbf{j}}$

For C, $\vec{v}_C = v\cos 60°\hat{\mathbf{i}} + v\sin 60°\hat{\mathbf{j}}$

Hence $\vec{v}_{\mathrm{BC}} = -2v\sin 60°\hat{\mathbf{j}} = -\sqrt{3}v\hat{\mathbf{j}}$

The speed of B as observed by C is $\sqrt{3}v \approx 996$ m/s ………… (30)

Notice that the relative velocities for each pair are anti-parallel.

Alternative solution for 1.4

One can obtain v_{BC} by considering the rotation about the axis at one of the spacecrafts.

$$v_{\mathrm{BC}} = \omega L = \frac{2\pi}{365 \times 24 \times 60 \times 60\ \mathrm{s}}(5 \times 10^6\ \mathrm{km}) \approx 996\ \mathrm{m/s}$$

2. An Electrified Soap Bubble

A spherical soap bubble with internal air density ρ_i, temperature T_i and radius R_0 is surrounded by air with density ρ_a, atmospheric pressure P_a and temperature T_a. The soap film has surface tension γ, density ρ_s and thickness t. The mass and the surface tension of the soap do not change with the temperature. Assume that $R_0 \gg t$.

The increase in energy, dE, that is needed to increase the surface area of a soap-air interface by dA, is given by $dE = \gamma dA$ where γ is the surface tension of the film.

2.1 Find the ratio $\dfrac{\rho_i T_i}{\rho_a T_a}$ in terms of γ, P_a and R_0. **[1.7 point]**

2.2 Find the numerical value of $\dfrac{\rho_i T_i}{\rho_a T_a} - 1$ using $\gamma = 0.0250\,\mathrm{Nm^{-1}}$, $R_0 = 1.00\ \mathrm{cm}$, and $P_a = 1.013\times 10^5\ \mathrm{Nm^{-2}}$. **[0.4 point]**

2.3 The bubble is initially formed with warmer air inside. Find the minimum numerical value of T_i such that the bubble can float in still air. Use $T_a = 300\ \mathrm{K}$, $\rho_s = 1000\ \ \mathrm{kgm^{-3}}$, $\rho_a = 1.30\ \ \mathrm{kgm^{-3}}$, $t = 100\ \mathrm{nm}$ and $g = 9.80\ \ \mathrm{ms^{-2}}$. **[2.0 points]**

After the bubble is formed for a while, it will be in thermal equilibrium with the surrounding. This bubble in still air will naturally fall towards the ground.

2.4 Find the minimum velocity u of an updraught (air flowing upwards) that will keep the bubble from falling at thermal equilibrium. Give your answer in terms of ρ_s, R_0, g, t and the air's coefficient of viscosity η. You may assume that the velocity is small such that Stokes's law applies, and ignore the change in the radius when the temperature lowers to the equilibrium. The drag force from Stokes' Law is $F = 6\pi\eta R_0 u$.

[1.6points]

2.5 Calculate the numerical value for u using $\eta = 1.8\times 10^{-5}\ \mathrm{kgm^{-1}\,s^{-1}}$. **[0.4 point]**

The above calculations suggest that the terms involving the surface tension γ add very little to the accuracy of the result. In all of the questions below, you can neglect the surface tension terms.

2.6 If this spherical bubble is now electrified uniformly with a total charge q, find an equation describing the new radius R_1 in terms of R_0, P_a, q and the permittivity of free space ε_0.

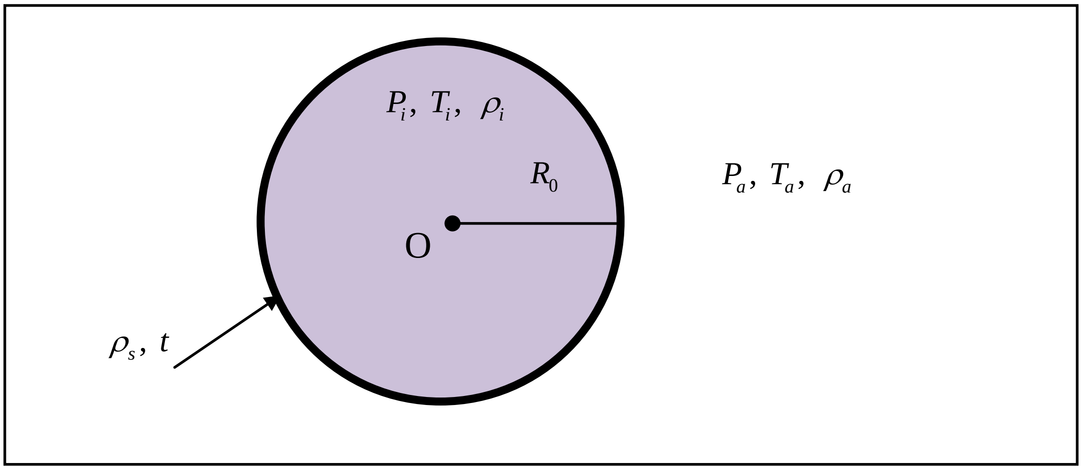
[2.0points]

2.7 Assume that the total charge is not too large (i.e. $\frac{q^2}{\varepsilon_0 R_0^4} << P_a$) and the bubble only experiences a small increase in its radius, find ΔR where $R_1 = R_0 + \Delta R$.
Given that $(1+x)^n \approx 1 + nx$ where $x \ll 1$. **[0.7 point]**

2.8 What must be the magnitude of this charge q in terms of $t, \rho_a, \rho_s, \varepsilon_0, R_0, P_a$ in order that the bubble will float motionlessly in still air? Calculate also the numerical value of q. The permittivity of free space $\varepsilon_0 = 8.85 \times 10^{-12}$ farad/m. **[1.2 point]**

2. SOLUTION

2.1. The bubble is surrounded by air.



Cutting the sphere in half and using the projected area to balance the forces give

$$P_i \pi R_0^2 = P_a \pi R_0^2 + 2\left(2\pi R_0 \gamma\right)$$
$$P_i = P_a + \frac{4\gamma}{R_0} \qquad \ldots (1)$$

The pressure and density are related by the ideal gas law:

$PV = nRT$ or $P = \frac{\rho RT}{M}$, where M = the molar mass of air. … (2)

Apply the ideal gas law to the air inside and outside the bubble, we get

$$\rho_i T_i = P_i \frac{M}{R}$$
$$\rho_a T_a = P_a \frac{M}{R},$$

$$\frac{\rho_i T_i}{\rho_a T_a} = \frac{P_i}{P_a} = \left[1 + \frac{4\gamma}{R_0 P_a}\right] \qquad \ldots (3)$$

2.2. Using $\gamma = 0.025\,\mathrm{Nm^{-1}}$, $R_0 = 1.0$ cm and $P_a = 1.013\times10^5\,\mathrm{Nm^{-2}}$, the numerical value of the ratio is

$$\frac{\rho_i T_i}{\rho_a T_a} = 1 + \frac{4\gamma}{R_0 P_a} = 1 + 0.0001 \qquad \dots (4)$$

(The effect of the surface tension is very small.)

2.3. Let W = total weight of the bubble, F = buoyant force due to air around the bubble

$$\begin{aligned} W &= (\text{mass of film+mass of air})\,g \\ &= \left(4\pi R_0^2\,\rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right) g \qquad \dots (5) \\ &= 4\pi R_0^2\,\rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right] g \end{aligned}$$

The buoyant force due to air around the bubble is

$$B = \frac{4}{3}\pi R_0^3 \rho_a g \qquad \dots (6)$$

If the bubble floats in still air,

$$B \ge W$$

$$\frac{4}{3}\pi R_0^3 \rho_a g \ge 4\pi R_0^2\,\rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right] g \qquad \dots (7)$$

Rearranging to give

$$\begin{aligned} T_i &\ge \frac{R_0 \rho_a T_a}{R_0 \rho_a - 3\rho_s t}\left[1 + \frac{4\gamma}{R_0 P_a}\right] \qquad \dots (8) \\ &\ge 307.1\ \mathrm{K} \end{aligned}$$

The air inside must be about $7.1^\circ\mathrm{C}$ warmer.

2.4. Ignore the radius change → Radius remains $R_0 = 1.0$ cm

(The radius actually decreases by 0.8% when the temperature decreases from 307.1 K to 300 K. The film itself also becomes slightly thicker.)

The drag force from Stokes' Law is $F = 6\pi\eta R_0 u$ … (9)

If the bubble floats in the updraught,

$$F \geq W - B$$

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2 \rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right) g - \frac{4}{3}\pi R_0^3 \rho_a g \qquad \text{… (10)}$$

When the bubble is in thermal equilibrium $T_i = T_a$.

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2 \rho_s t + \frac{4}{3}\pi R_0^3 \rho_a \left[1 + \frac{4\gamma}{R_0 P_a}\right]\right) g - \frac{4}{3}\pi R_0^3 \rho_a g$$

Rearranging to give

$$u \geq \frac{4R_0 \rho_s t g}{6\eta} + \frac{\frac{4}{3} R_0^2 \rho_a g \left(\frac{4\gamma}{R_0 P_a}\right)}{6\eta} \qquad \text{… (11)}$$

2.5. The numerical value is $u \geq 0.36$ m/s.

The 2nd term is about 3 orders of magnitude lower than the 1st term.

From now on, ignore the surface tension terms.

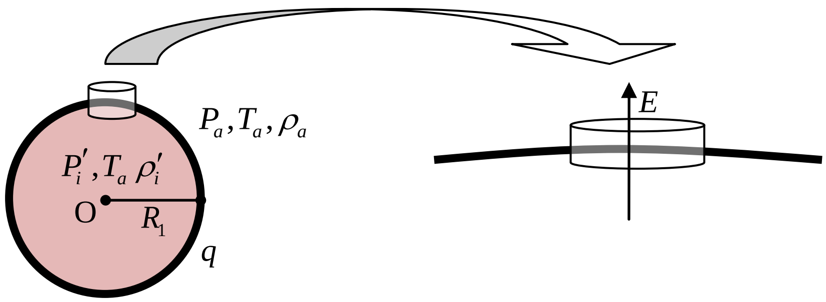
2.6. When the bubble is electrified, the electrical repulsion will cause the bubble to expand in size and thereby raise the buoyant force.

The force/area is (e-field on the surface × charge/area)

There are two alternatives to calculate the electric field ON the surface of the soap film.

A. From Gauss's Law

Consider a very thin pill box on the soap surface.



E = electric field on the film surface that results from all other parts of the soap film, excluding the surface inside the pill box itself.

E_q = total field just outside the pill box = $\dfrac{q}{4\pi\varepsilon_0 R_1^2} = \dfrac{\sigma}{\varepsilon_0}$

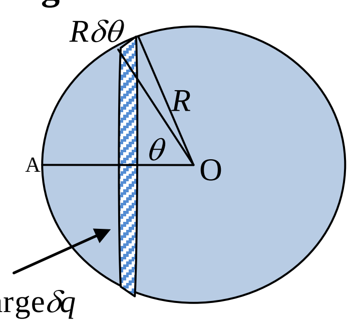
= E + electric field from surface charge σ

= $E + E_\sigma$

Using Gauss's Law on the pill box, we have $E_\sigma = \dfrac{\sigma}{2\varepsilon_0}$ perpendicular to the film as a result of symmetry.

Therefore, $E = E_q - E_\sigma = \dfrac{\sigma}{\varepsilon_0} - \dfrac{\sigma}{2\varepsilon_0} = \dfrac{\sigma}{2\varepsilon_0} = \dfrac{1}{2\varepsilon_0}\dfrac{q}{4\pi R_1^2}$ … (12)

B. From direct integration



$$\delta q = \left(\frac{q}{4\pi R^2}\right) 2\pi R \sin\theta . R\delta\theta$$

To find the magnitude of the electrical repulsion we must first find the electric field intensity E at a point on (not outside) the surface itself.

Field at A in the direction $\overrightarrow{\mathrm{OA}}$ is

$$\delta E_A = \frac{1}{4\pi\varepsilon_0}\frac{\left(q/4\pi R_1^2\right)2\pi R_1^2\sin\theta\delta\theta}{\left(2R_1\sin\frac{\theta}{2}\right)^2}\sin\frac{\theta}{2} = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0}\cos\frac{\theta}{2}\delta\left(\frac{\theta}{2}\right)$$

$$E_A = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0}\int_{\theta=0}^{\theta=180^\circ}\cos\frac{\theta}{2}d\left(\frac{\theta}{2}\right) = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0} \quad \ldots (13)$$

The repulsive force per unit area of the surface of bubble is

$$\left(\frac{q}{4\pi R_1^2}\right)E = \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} \quad \ldots (14)$$

Let P_i' and ρ_i' be the new pressure and density when the bubble is electrified.

This electric repulsive force will augment the gaseous pressure P_i'.

P_i' is related to the original P_i through the gas law.

$$P_i'\frac{4}{3}\pi R_1^3 = P_i\frac{4}{3}\pi R_0^3$$

$$P_i' = \left(\frac{R_0}{R_1}\right)^3 P_i = \left(\frac{R_0}{R_1}\right)^3 P_a \quad \ldots (15)$$

<u>In the last equation, the surface tension term has been ignored.</u>

From balancing the forces on the half-sphere projected area, we have (again ignoring the surface tension term)

$$P_i' + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} = P_a$$

$$P_a\left(\frac{R_0}{R_1}\right)^3 + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} = P_a \quad \ldots (16)$$

Rearranging to get

$$\left(\frac{R_1}{R_0}\right)^4 - \left(\frac{R_1}{R_0}\right) - \frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} = 0 \qquad \dots (17)$$

Note that (17) yields $\frac{R_1}{R_0} = 1$ when $q = 0$, as expected.

2.7. <u>Approximate solution</u> for R_1 when $\frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} << 1$

Write $R_1 = R_0 + \Delta R,\ \Delta R << R_0$

Therefore, $\frac{R_1}{R_0} = 1 + \frac{\Delta R}{R_0},\ \left(\frac{R_1}{R_0}\right)^4 \approx 1 + 4\frac{\Delta R}{R_0}$ … (18)

Eq. (17) gives:

$$\Delta R \approx \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \qquad \dots (19)$$

$$R_1 \approx R_0 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \approx R_0\left(1 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^4 P_a}\right) \qquad \dots (20)$$

2.8. The bubble will float if

$$B \geq W$$

$$\frac{4}{3}\pi R_1^3 \rho_a g \geq 4\pi R_0^2\, \rho_s t g + \frac{4}{3}\pi R_0^3 \rho_i g \qquad \dots (21)$$

Initially, $T_i = T_a \Rightarrow \rho_i = \rho_a$ for $\gamma \to 0$ and $R_1 = R_0\left(1 + \frac{\Delta R}{R_0}\right)$

$$\frac{4}{3}\pi R_0^3\left(1+\frac{\Delta R}{R_0}\right)^3 \rho_a g \geq 4\pi R_0^2\,\rho_s t g + \frac{4}{3}\pi R_0^3 \rho_a g$$

$$\frac{4}{3}\pi\left(3\Delta R\right)\rho_a g \geq 4\pi R_0^2\,\rho_s t g$$

$$\frac{4}{3}\pi\frac{3q^2}{96\pi^2\varepsilon_0 R_0 P_a}\rho_a g \geq 4\pi R_0^2\,\rho_s t g \qquad \ldots (22)$$

$$q^2 \geq \frac{96\pi^2 R_0^3\,\rho_s t\varepsilon_0 P_a}{\rho_a}$$

$$q \;\approx\; 256\times10^{-9}\ \text{C} \;\approx 256\,\text{nC}$$

Note that if the surface tension term is retained, we get

$$R_1 \approx \left(1+\frac{q^2/96\pi^2\varepsilon_0 R_0^4 P_a}{\left[1+\frac{2}{3}\left(\frac{4\gamma}{R_0 P_a}\right)\right]}\right)R_0$$

3. To Commemorate the Centenary of Rutherford's Atomic Nucleus: the Scattering of an Ion by a Neutral Atom

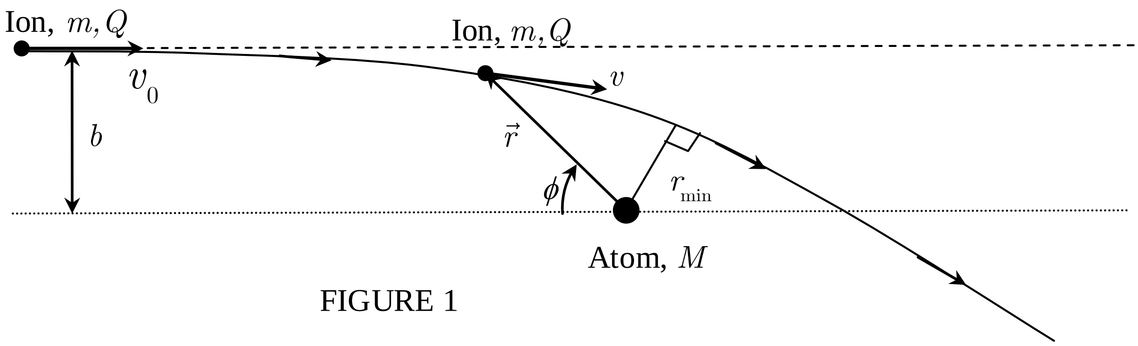


FIGURE 1

An ion of mass m, charge Q, is moving with an initial non-relativistic speed v_0 from a great distance towards the vicinity of a neutral atom of mass $M >> m$ and of electrical polarisability α. The impact parameter is b as shown in Figure 1.

The atom is instantaneously polarised by the electric field $\vec{E}$ of the in-coming (approaching) ion. The resulting electric dipole moment of the atom is $\vec{p} = \alpha\vec{E}$. Ignore any radiative losses in this problem.

3.1 Calculate the electric field intensity $\vec{E}_p$ at a distance r from an ideal electric dipole $\vec{p}$ at the origin O along the direction of $\vec{p}$ in Figure 2. **[1.2 points]**

$p = 2aq, \quad r \gg a$

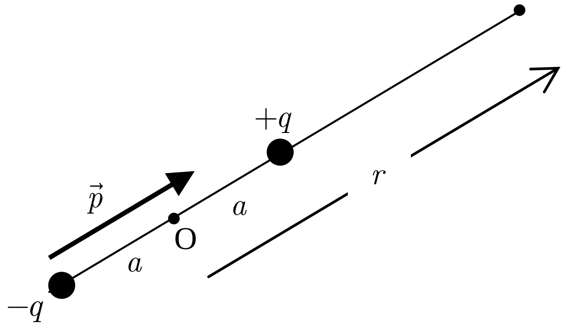


FIGURE 2

3.2 Find the expression for the force $\vec{f}$ acting on the ion due to the polarised atom. Show that this force is attractive regardless of the sign of the charge of the ion.

[3.0 points]

3.3 What is the electric potential energy of the ion-atom interaction in terms of α, Q and r?

[0.9 points]

3.4 Find the expression for $r_{\min}$, the distance of the closest approach, as shown in Figure 1.

[2.4 points]

3.5 If the impact parameter b is less than a critical value b_0, the ion will descend along a spiral to the atom. In such a case, the ion will be neutralized, and the atom is, in turn, charged. This process is known as the "charge exchange" interaction. What is the cross sectional area $A = \pi b_0^2$ of this "charge exchange" collision of the atom as seen by the ion? **[2.5 points]**

QUESTION 3: SOLUTION

1. Using Coulomb's Law, we write the electric field at a distance r is given by

$$E_p = \frac{q}{4\pi\varepsilon_0 (r-a)^2} - \frac{q}{4\pi\varepsilon_0 (r+a)^2}$$

$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{1}{\left(1-\frac{a}{r}\right)^2} - \frac{1}{\left(1+\frac{a}{r}\right)^2}\right) \qquad \text{..................(1)}$$

Using binomial expansion for small a,

$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(1+\frac{2a}{r}-1+\frac{2a}{r}\right)$$

$$= +\frac{4qa}{4\pi\varepsilon_0 r^3} = +\frac{qa}{\pi\varepsilon_0 r^3} \qquad \text{................(2)}$$

$$= \frac{2p}{4\pi\varepsilon_0 r^3}$$

2. The electric field seen by the atom from the ion is

$$\vec{E}_{ion} = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \text{................ (3)}$$

The induced dipole moment is then simply

$$\vec{p} = \alpha\vec{E}_{ion} = -\frac{\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \text{................ (4)}$$

From eq. (2)

$$\vec{E}_p = \frac{2p}{4\pi\varepsilon_0 r^3}\hat{r}$$

The electric field intensity $\vec{E}_p$ at the position of an ion at that instant is, using eq. (4),

$$\vec{E}_p = \frac{1}{4\pi\varepsilon_0 r^3}\left[-\frac{2\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r}\right] = -\frac{\alpha Q}{8\pi^2\varepsilon_0^2 r^5}\hat{r}$$

The force acting on the ion is

$$\vec{f} = Q\vec{E}_p = -\frac{\alpha Q^2}{8\pi^2\varepsilon_0^2 r^5}\hat{r} \qquad \text{................ (5)}$$

The "-'' sign implies that this force is attractive and Q^2 implies that the force is attractive regardless of the sign of Q.

3. The potential energy of the ion-atom is given by $U = \int_r^\infty \vec{f}.d\vec{r}$ ……….……………………….(6)

Using this, $U = \int_r^\infty \vec{f}.d\vec{r} = -\dfrac{\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4}$ ……………………………………………………….(7)

[Remark: Students might use the term $-\vec{p}\cdot\vec{E}$ which changes only the factor in front.]

4. At the position $r_{\min}$ we have, according to the Principle of Conservation of Angular Momentum,

$$mv_{\max}r_{\min} = mv_0 b$$

$$v_{\max} = v_0\frac{b}{r_{\min}} \qquad \text{…………….. (8)}$$

And according to the Principle of Conservation of Energy:

$$\frac{1}{2}mv_{\max}^2 + \frac{-\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4} = \frac{1}{2}mv_0^2 \qquad \text{…………….. (9)}$$

Eqs.(12) & (13):

$$\left(\frac{b}{r_{\min}}\right)^2 - \frac{\alpha Q^2 \Big/ \frac{1}{2}mv_0^2}{32\pi^2\varepsilon_0^2 b^4}\left(\frac{b}{r_{\min}}\right)^4 = 1$$

$$\left(\frac{r_{\min}}{b}\right)^4 - \left(\frac{r_{\min}}{b}\right)^2 + \frac{\alpha Q^2}{16\pi^2\varepsilon_0^2 mv_0^2 b^4} = 0 \qquad \text{…………….. (10)}$$

The roots of eq. (14) are:

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 \pm \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 mv_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{…………….. (11)}$$

[Note that the equation (14) implies that $r_{\min}$ cannot be zero, unless b is itself zero.]

Since the expression has to be valid at $Q = 0$, which gives

$$r_{\min} = \frac{b}{\sqrt{2}}[1 \pm 1]^{\frac{1}{2}}$$

We have to choose "+" sign to make $r_{\min} = b$

Hence,

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 + \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 mv_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{……………………………………......(12)}$$

5. A spiral trajectory occurs when (16) is imaginary (because there is no minimum distance of approach).

$r_{\min}$ is real under the condition:

$$1 \geq \frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2 b^4}$$

$$b \geq b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}} \qquad \text{……………..} (13)$$

For $b < b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}}$ the ion will collide with the atom.

Hence the atom, as seen by the ion, has a cross-sectional area A,

$$A = \pi b_0^2 = \pi \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{2}} \qquad \text{……………..} (14)$$

1. Electrical Blackbox: Capacitive Displacement Sensor

For a capacitor of capacitance C which is a component of a relaxation oscillator whose frequency of oscillation is f, the relationship between f and C is as follows:

$$f = \frac{\alpha}{C + C_S}$$

where α is a constant and C_S is the stray capacitance of our circuits. The frequency f can be monitored using a digital frequency meter.

The electrical blackbox given in this experiment is a parallel plate capacitor. Each plate consists of a number of small teeth of the same geometrical shape. The value of C can be varied by displacing the upper plate relative to the lower plate, horizontally. Between the two plates there is a sheet of dielectric material.

Equipment: a relaxation oscillator, a digital multimeter for measuring frequency of the relaxation oscillator, a set of capacitors of known capacitances, an electrical blackbox and a battery.

Caution: Check the voltage of the battery and ask for a new one if the voltage is less than 9 V. Do not forget to switch on.

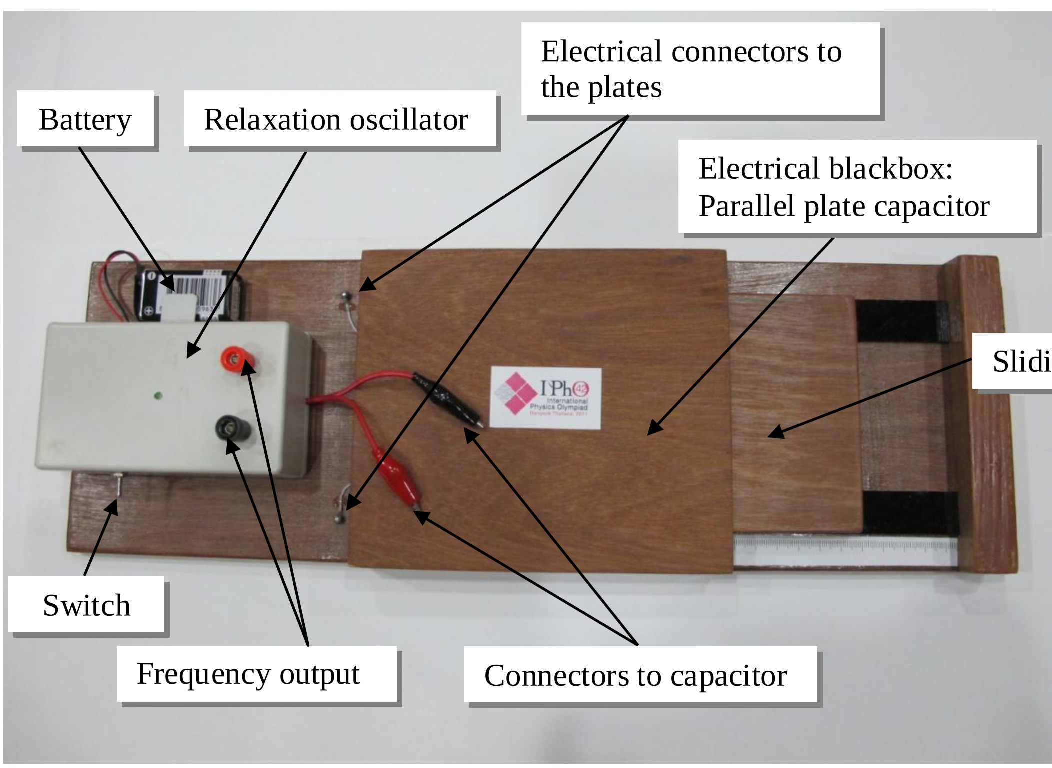


FIGURE 1

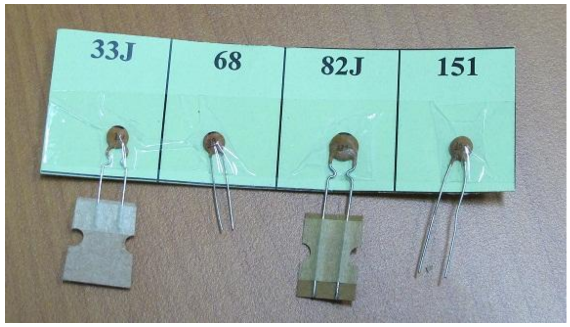


FIGURE 2 Capacitors

The position for frequency measurements

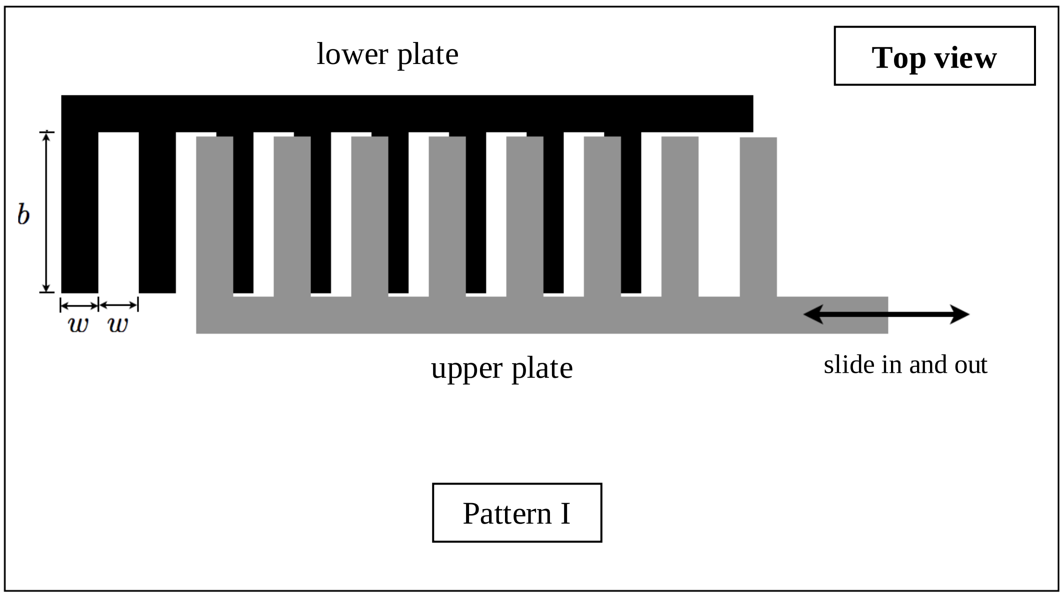
FIGURE 3 Digital multimeter for measuring frequency

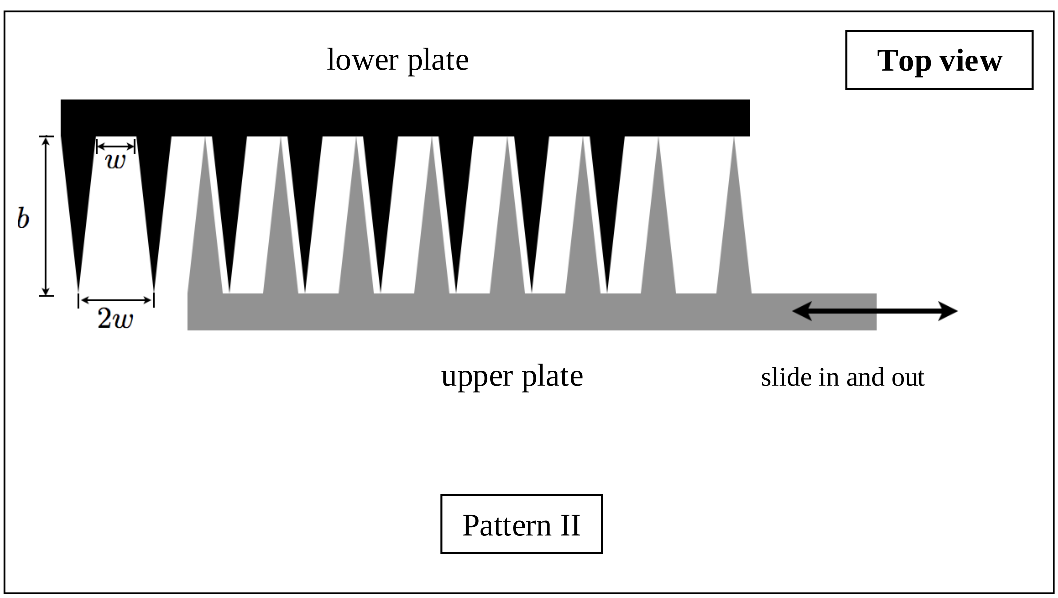
TABLE 1 Nominal Capacitance values

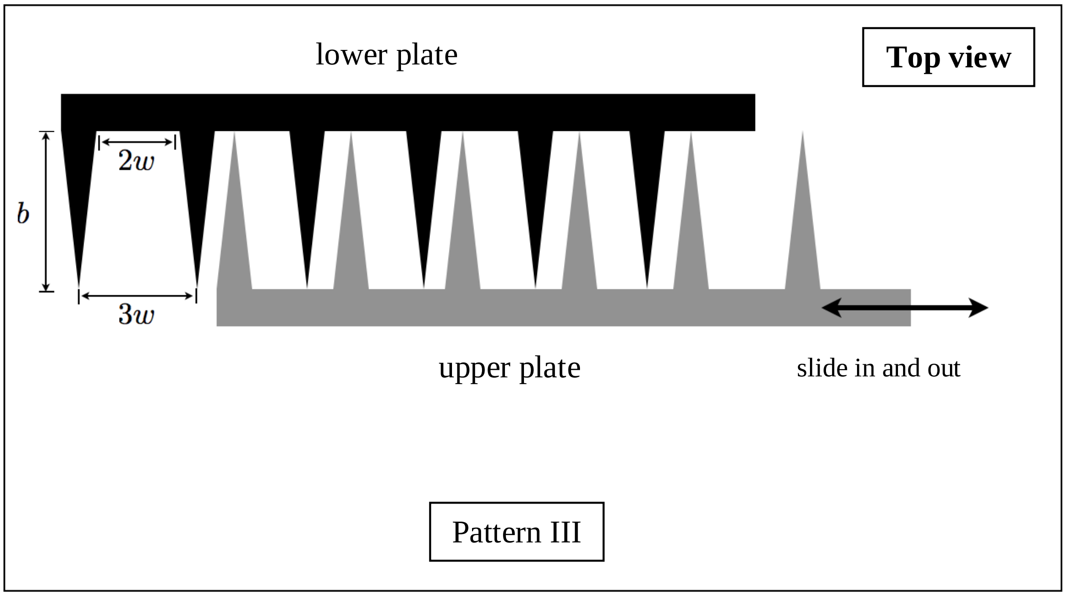
Code	Capacitance value (pF)
33J	34 ± 1
68	68 ± 1
82J	84 ± 1
151	150 ± 1

Part 1. Calibration
Perform the measurement of f using the given capacitors of known capacitances. Draw appropriate graph to find the value of α and C_S. Error analysis is not required. **[3.0 points]**

Part 2. Determination of geometrical shape of a parallel plate capacitor **[6.0 points]**
Given the three possible geometrical shapes as Pattern I, Pattern II and Pattern III as follows:







For each pattern, draw qualitatively an expected graph of C versus the positions of the upper plate but label the x-axis. Then, perform the measurement of f versus the positions of the upper plate. Plot graphs and, from these graphs, deduce the pattern of the parallel plate capacitor and its dimensions (values of b and w). The separation d between the upper and lower plates is 0.20 mm. The dielectric sheet between the plates has a dielectric constant $K = 1.5$. The permittivity of free space $\varepsilon_0 = 8.85 \times 10^{-12}$ $\mathrm{Fm^{-1}}$. Error analysis is not required.

Part 3. Resolution of digital calipers **[1.0 point]**

As the relative position of the parallel plates is varied, the capacitance changes with a pattern. This set-up may be used as digital calipers for measuring length. If the parallel plate capacitor in this experiment is to be used as digital calipers, estimate from the experimental data in Part 2 its resolution: the smallest distance that can be measured for the frequency value $f \approx 5\ \mathrm{kHz}$. An error estimate for the final answer is not required.

Part 1. Calibration

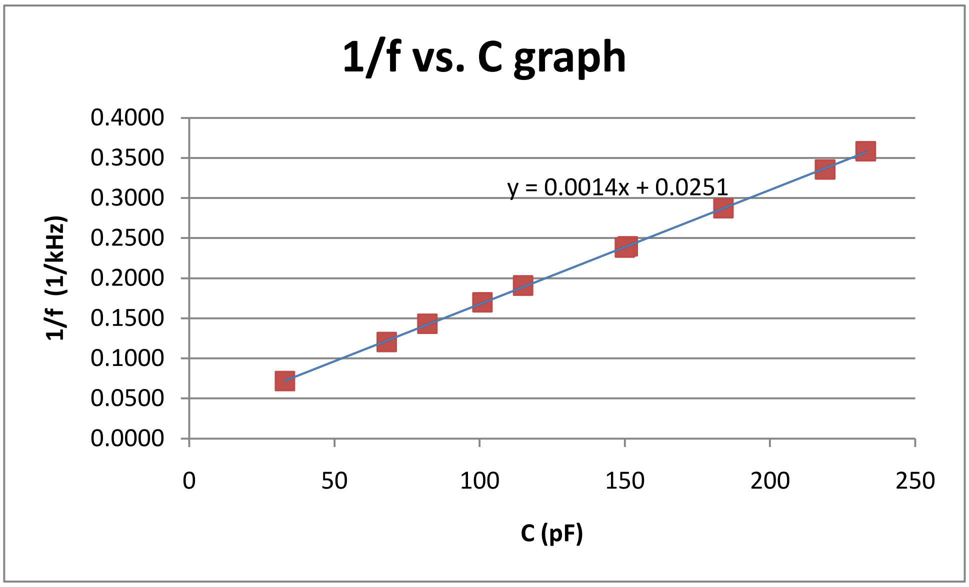
From the relationship between f and C given,

$$f = \frac{\alpha}{C + C_s} \qquad \Leftrightarrow \qquad \frac{1}{f} = \frac{1}{\alpha}C + \frac{C_s}{\alpha}$$

That is, theoretically, the graph of $\frac{1}{f}$ on the Y-axis versus C on the X-axis should be linear of which the slope and the Y-intercept is $\frac{1}{\alpha}$ and $\frac{C_s}{\alpha}$ respectively.

The table below shows the measured values of C (plotted on the X-axis,) f and, additionally, $\frac{1}{f}$, which is plotted on the Y-axis.

C (pF)	f (kHz)	1/f (ms)
33	13.94	0.0717
68	8.30	0.1205
82	6.99	0.1431
151	4.17	0.2398
233	2.79	0.3584
219	2.98	0.3356
184	3.48	0.2874
150	4.20	0.2381
115	5.24	0.1908
101	5.89	0.1698



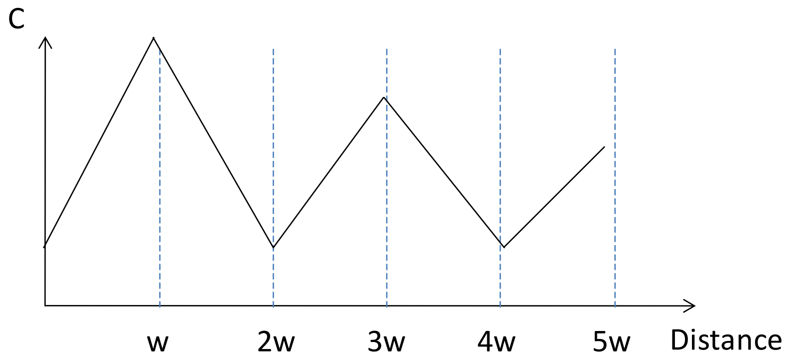
From this graph, the slope ($\frac{1}{\alpha}$) and the Y-intercept ($\frac{C_s}{\alpha}$) is equal to 0.0014 s/nF and 0.0251 ms respectively.

Hence, $$\alpha = \frac{1}{\text{slope}} = \frac{1}{0.0014\ \text{s}/\text{nF}} = 714\ \text{nF/s}$$

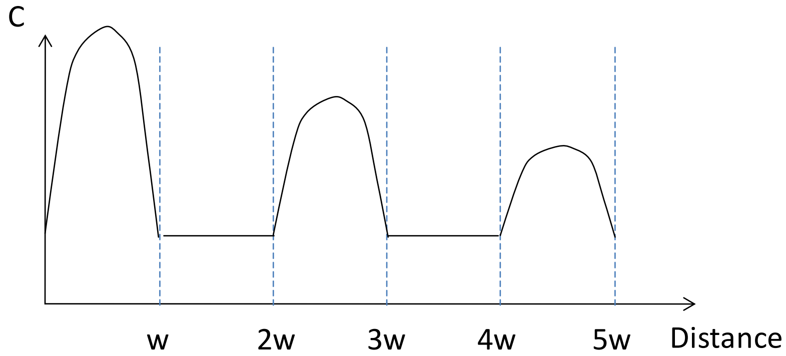
and $$C_s = \frac{\text{Y} - \text{intercept}}{\text{slope}} = \frac{0.0251\ \text{ms}}{0.0014\ \text{s}/\text{nF}} = 17.9\ \text{pF}$$ as required.

Part II. Determination of geometrical shape of parallel-plates capacitor

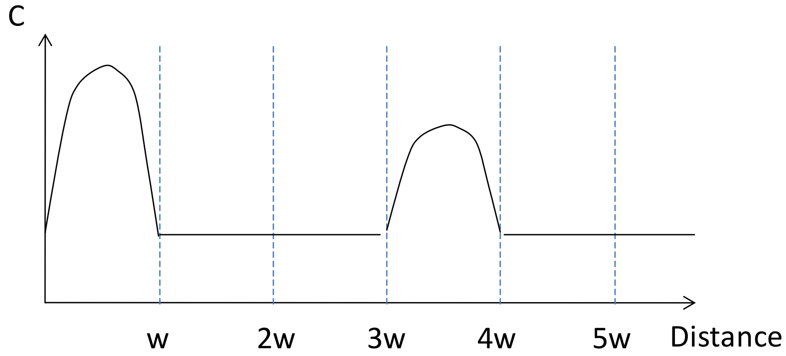
PATTERN I: The expected graph of C versus the position



PATTERN II: The expected graph of C versus the position

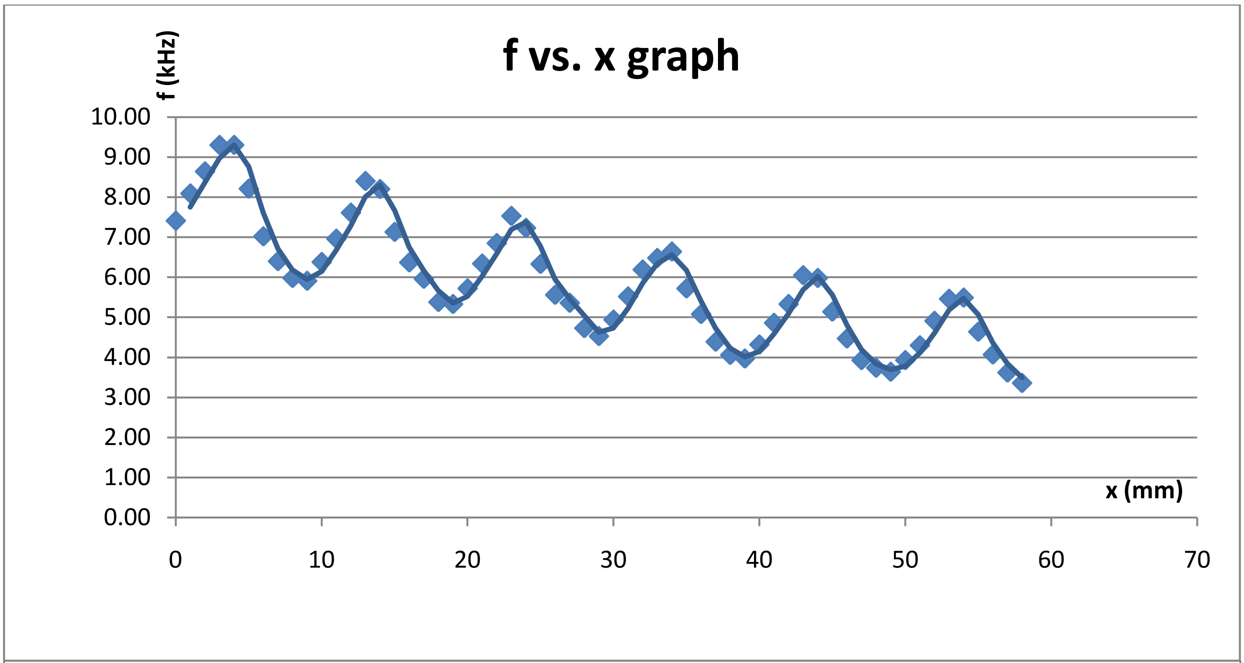


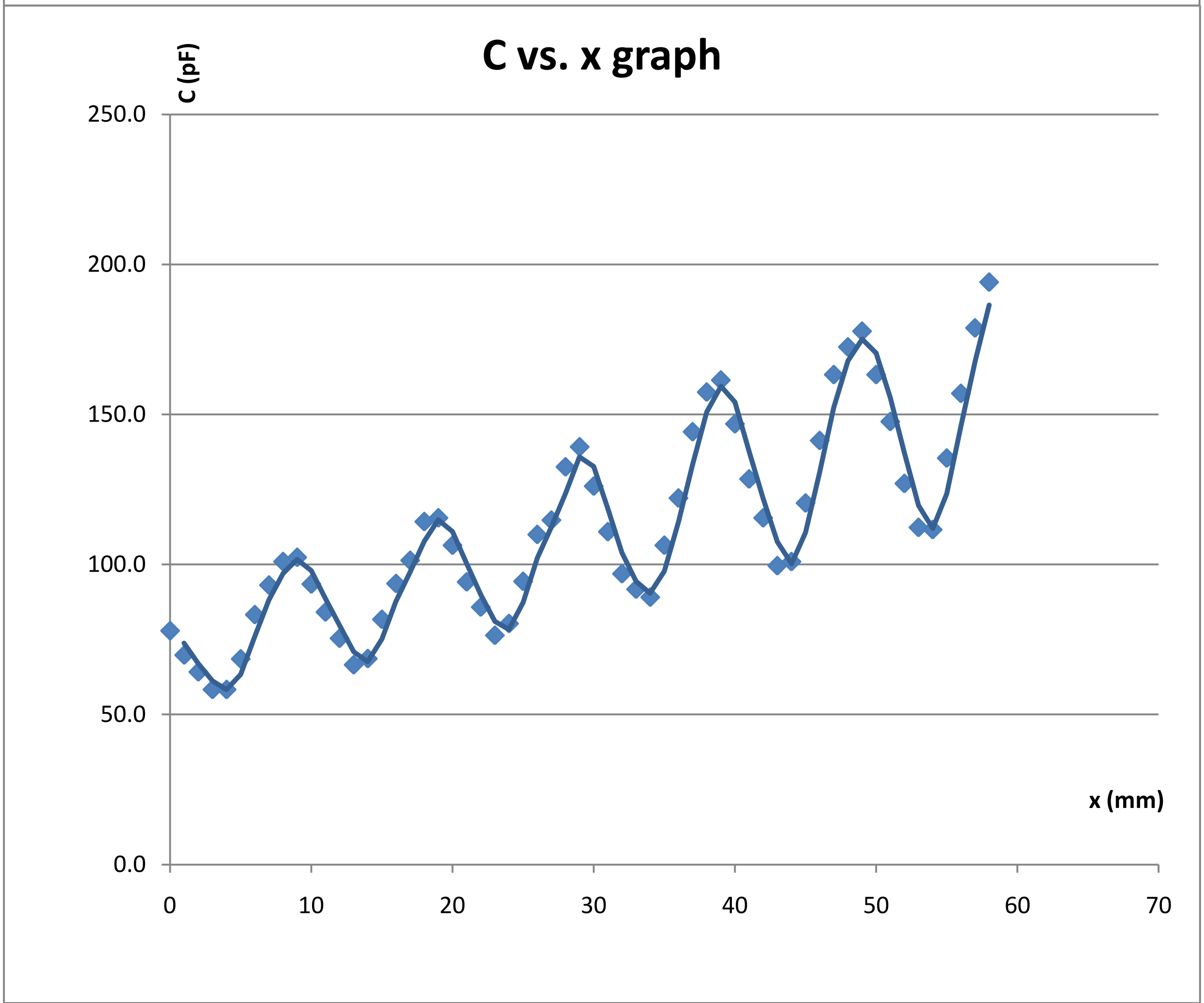
PATTERN III: The expected graph of C versus the position



By measuring f and C versus x (the distance moved between the two plates,) the data and the graphs are shown below.

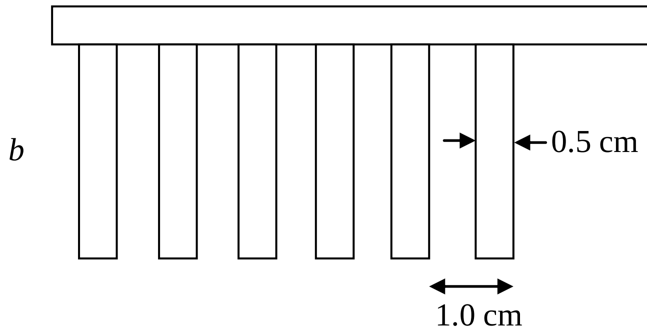
x (mm)	f (kHz)	C (pF)	x (mm)	f (kHz)	C (pF)
0	7.41	77.9	30	4.94	126.1
1	8.09	69.8	31	5.52	110.9
2	8.64	64.2	32	6.19	96.9
3	9.30	58.3	33	6.48	91.7
4	9.30	58.3	34	6.64	89.1
5	8.21	68.5	35	5.72	106.4
6	7.02	83.3	36	5.08	122.1
7	6.40	93.1	37	4.39	144.2
8	5.98	100.9	38	4.06	157.4
9	5.91	102.4	39	3.97	161.4
10	6.38	93.5	40	4.32	146.8
11	6.96	84.1	41	4.86	128.5
12	7.61	75.4	42	5.33	115.5
13	8.40	66.5	43	6.05	99.6
14	8.20	68.6	44	5.98	100.9
15	7.13	81.7	45	5.14	120.5
16	6.37	93.6	46	4.47	141.3
17	5.96	101.3	47	3.93	163.3
18	5.38	114.3	48	3.74	172.5
19	5.33	115.5	49	3.64	177.7
20	5.72	106.4	50	3.93	163.3
21	6.34	94.2	51	4.30	147.6
22	6.85	85.8	52	4.91	127.0
23	7.53	76.4	53	5.46	112.3
24	7.23	80.3	54	5.49	111.6
25	6.33	94.3	55	4.64	135.4
26	5.56	110.0	56	4.07	157.0
27	5.36	114.8	57	3.62	178.8
28	4.73	132.5	58	3.36	194.1
29	4.53	139.2			


f vs. x graph
f (kHz)
10.00
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
x (mm)
0
10
20
30
40
50
60
70


C vs. x graph
C (pF)
250.0
200.0
150.0
100.0
50.0
0.0
x (mm)
0
10
20
30
40
50
60
70

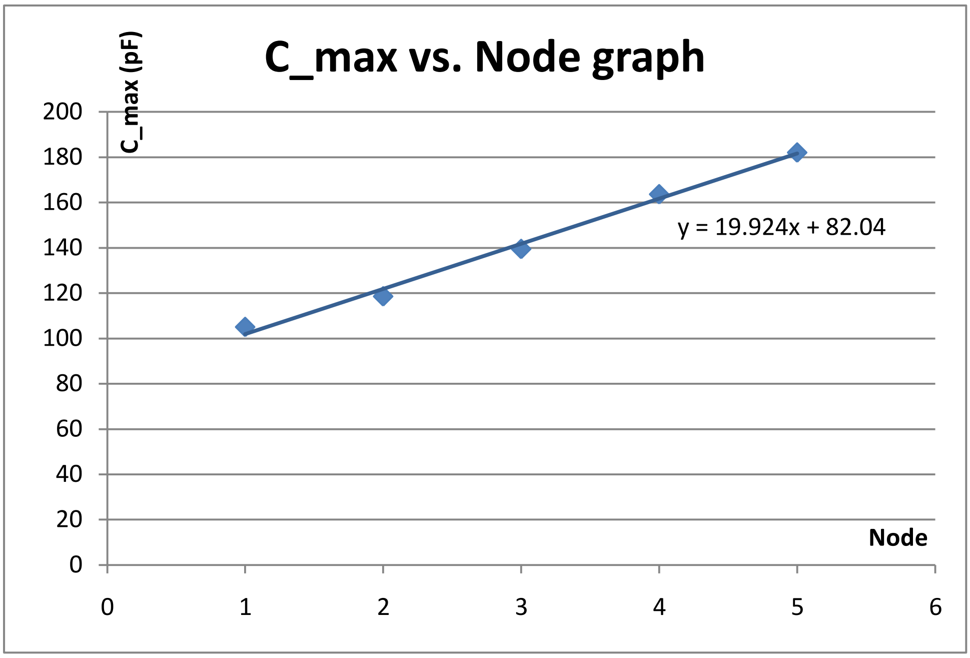
From periodicity of the graph, period = 1.0 cm

Simple possible configuration is:



The peaks of C values obtained from the C vs. x graph are provided in the table below.
These maximum C are plotted (on the Y-axis) vs. nodes (on the X-axis.)

node	C_max
1	105.1
2	118.6
3	139.5
4	163.7
5	182.1



This graph is linear of which the slope is the dropped off capacitance $\Delta C = 19.9$ pF/section.

Given that the distance between the plates $d = 0.20$ mm, $K = 1.5$,

$$\Delta C \approx \frac{K\varepsilon_0 A}{d},$$

and $A = 5\times10^{-3}\,\mathrm{m} \times b\,\mathrm{mm} \times 10^{-3}\ \mathrm{m}^2$

Then, $b\ \text{mm} \approx \dfrac{\Delta C\ d}{K\varepsilon_0 \times 10^{-3} \times 5 \times 10^{-3}} \approx 60\ \text{mm}$ if medium between plates is the dielectric of which $K = 1.5$.

Part III. Resolution of digital micrometer

From the given relationship between f and C, $f = \dfrac{\alpha}{C + C_S}$,

$$\Delta f \simeq \left|\frac{df}{dC}\right| \Delta C = \left|\frac{-\alpha}{(C + C_S)^2}\right| \Delta C$$

$$= \frac{f^2}{\alpha} \Delta C$$

$$\Leftrightarrow \qquad \Delta C = \frac{\alpha}{f^2} \Delta f$$

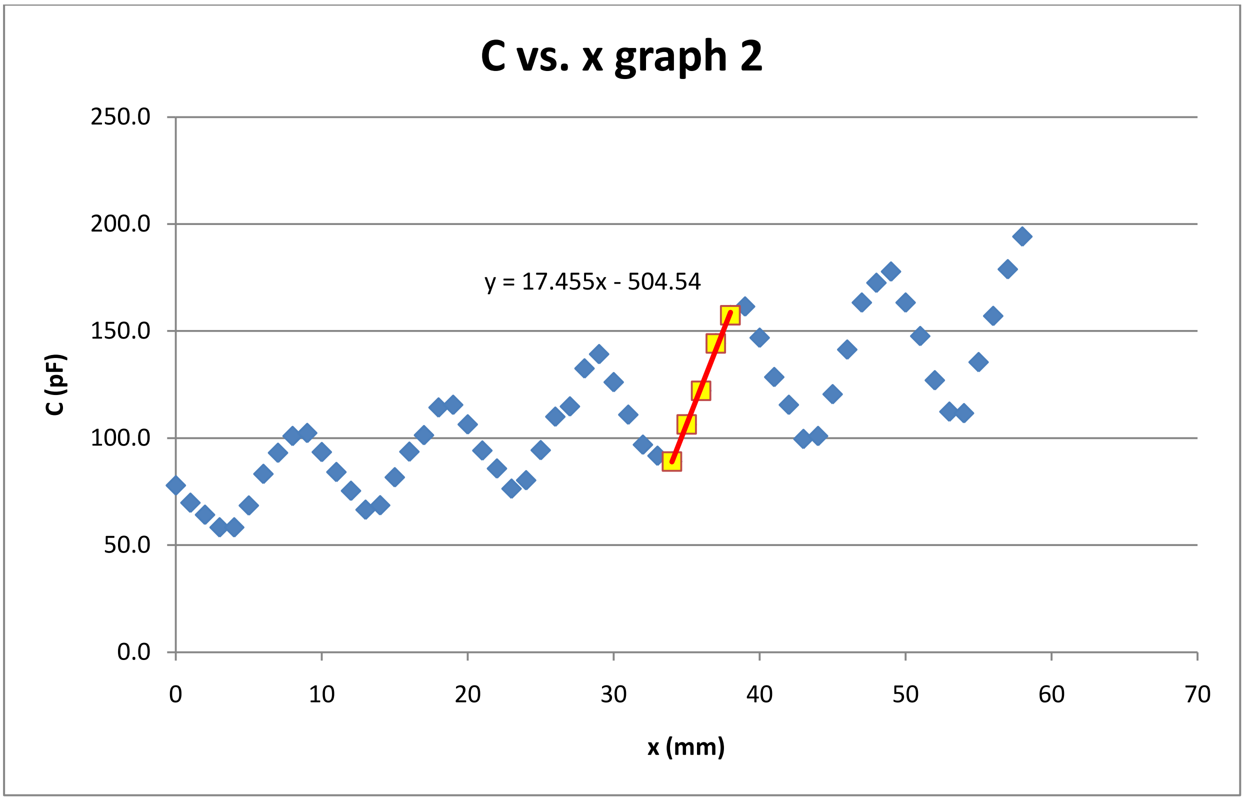
And since C linearly depends on x, $C = mx + \beta \quad \Rightarrow \quad \Delta C = m\Delta x$.

Hence,

$$\Delta x = \frac{\alpha}{mf^2} \Delta f,$$

where Δf is the smallest change of the frequency f which can be detected by the multimeter, x_0 is the operated distance at f = 5 kHz, and m is the gradient of the C vs. x graph at $x = x_0$.

From the f vs. x graph, at f = 5 kHz, The gradient is then measured on the C vs. x graph around this range.



From this graph, $m = 17.5\ \mathrm{pF/mm} = 1.75\times10^{-8}\,\mathrm{F/m}$.
Using this value of m, $f = 5\ \mathrm{kHz}$, $\alpha = 714\ \mathrm{nF/s}$, and $\Delta f = 0.01\ \mathrm{kHz}$,

$$\Delta x = \frac{714\times10^{-9}}{(1.75\times10^{-8})(5\times10^{3})^{2}}\times(0.01\times10^{3}) = 0.016\,\mathrm{mm}$$

NB. The C vs. x graph is used since C (but not f) is linearly related to x.

Alternative method for finding the resolution
(not strictly correct)

Using the f vs. x graph and the data in the table around $f = 5\ \mathrm{kHz}$, it is found that when f is changed by 1 kHz ($\Delta f = 1\ \mathrm{kHz}$,) x is roughly changed by 1.5 mm ($\Delta x \simeq 1.5\ \mathrm{mm}$.) Hence, when f is changed by $\Delta f = 0.01\ \mathrm{kHz}$ (the smallest detectable of the change,) the distance moved is $\Delta x \simeq 0.015\ \mathrm{mm}$.

2. Mechanical Blackbox: a cylinder with a ball inside

A small massive particle (ball) of mass m is fixed at distance z below the top of a long hollow cylinder of mass M. A series of holes are drilled perpendicularly to the central axis of the cylinder. These holes are for pivoting so that the cylinder will hang in a vertical plane.

Students are required to perform necessary nondestructive measurements to determine the numerical values of the following with their error estimates:

i. position of centre of mass of cylinder with ball inside.

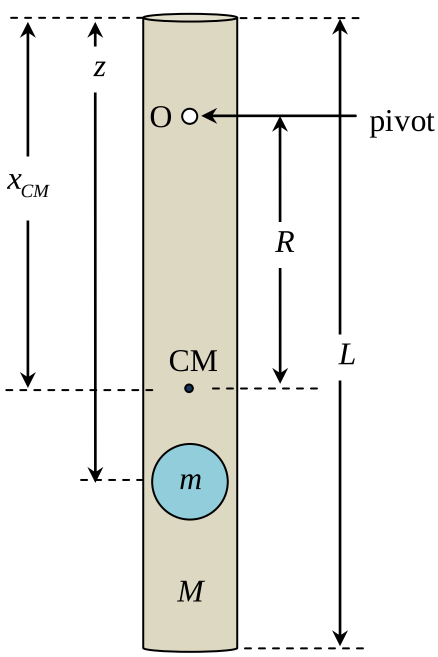
   Also provide a schematic drawing of the experimental set-up for measuring the centre of mass. **[1.0 points]**

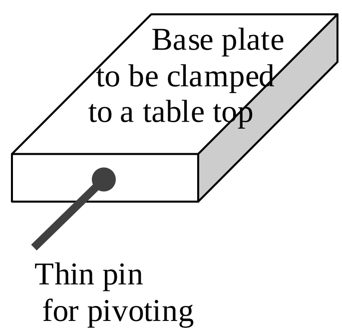
ii. distance z **[3.5 points]**

iii. ratio $\dfrac{M}{m}$. **[3.5 points]**

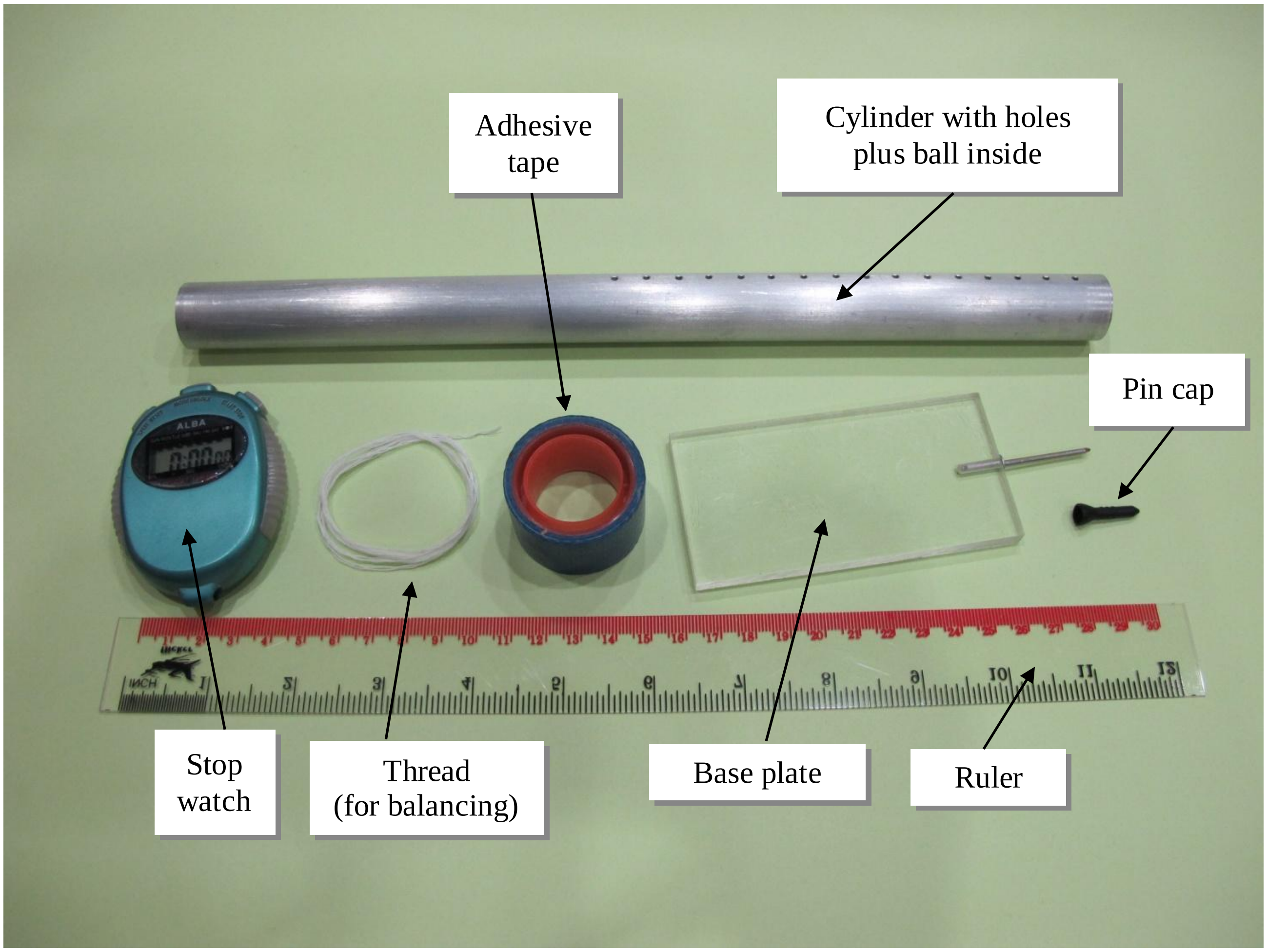
iv. the acceleration due to gravity, g. **[2.0 points]**

Equipment: a cylinder with holes plus a ball inside, a base plate with a thin pin, a pin cap, a ruler, a stop watch, thread, a pencil and adhesive tape.





x_{CM} is the distance from the top of the cylinder to the centre of mass.
R is the distance from the pivoting point to the centre of mass.

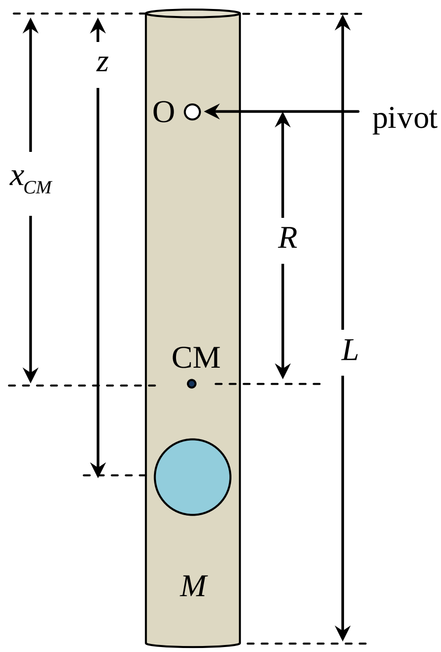


Caution: The thin pin is sharp. When it is not in use, it should be protected with a pin cap for safety.

Useful information:

1. For such a physical pendulum, $M+m\ R^2+I_{CM}\ \dfrac{d^2\theta}{dt^2}\ \approx -g\ M+m\ R\theta$, where I_{CM} is the moment of inertia of the cylinder with a ball about the centre of mass and θ is the angular displacement.
2. For a long hollow cylinder of length L and mass M, the moment of inertia about the centre of mass with the rotational axis perpendicular to the cylinder can be approximated by $\dfrac{1}{3}M\left(\dfrac{L}{2}\right)^2$.
3. The parallel axis theorem: $I = I_{\text{centre of mass}} + \mathfrak{M}x^2$, where x is the distance from the rotation point to the centre of mass, and $\mathfrak{M}$ is the total mass of the object.
4. The ball can be treated as a point mass and it is located on the central axis of the cylinder.
5. Assume that the cylinder is uniform and the mass of the end-caps is negligible.

Solution: 2 . Mechanical Blackbox: a cylinder with a ball inside



In order to be able to calculate the required values in i, ii, iii, we need to know:

a. the position of the centre of mass of the tubing plus particle (object) which depends on z, m, M

b. the moment of inertia of the above.

The position of the CM may be found by balancing. The I_{CM} can be calculated from the period of oscillation of the tubing plus object.

Analytical steps to select parameters for plotting

I. $$x_{CM} = \frac{mz + M\frac{L}{2}}{m+M} \quad \text{……………………. (1)}$$

L is readily obtainable with a ruler.

x_{CM} is determined by balancing the tubing and object.

II. For small-amplitude oscillation about any point O the period T is given by considering the equation:

$$\left\{(M+m)R^2+I_{CM}\right\}\ddot{\theta} = -g(M+m)R\sin\theta \approx -g(M+m)R\theta \quad \text{................ (2)}$$

$$T = 2\pi\sqrt{\frac{I_{CM}+(M+m)R^2}{g(M+m)R}} \quad \text{........................ (3)}$$

where
$$I_{CM} = \frac{1}{3}M\left(\frac{L}{2}\right)^2+M\left(x_{CM}-\frac{L}{2}\right)^2+m(z-x_{CM})^2$$
$$= \frac{1}{3}ML^2+Mx_{CM}^2-MLx_{CM}+m(z-x_{CM})^2 \quad \text{........................ (4)}$$

Note that

$$T^2\frac{g(M+m)}{4\pi^2} = \frac{I_{CM}}{R}+(M+m)R \quad \text{........................ (5)}$$

Method (a): (linear graph method)

The equation (5) may be put in the form:

$$T^2R = \left(\frac{4\pi^2}{g}\right)R^2+\frac{4\pi^2 I_{CM}}{(M+m)g} \quad \text{........................ (6)}$$

Hence the plot of T^2R v.s. R^2 will yield the straight line whose

Slope $\alpha = \dfrac{4\pi^2}{g}$ .... (7)

and y-intercept $\beta = \dfrac{4\pi^2 I_{CM}}{(M+m)g}$ .... (8)

Hence, $I_{CM} = (M+m)\dfrac{\beta}{\alpha}$ .... (9)

The value of g is from equation (7): $g = \dfrac{4\pi^2}{\alpha}$ .... (10)

Method (b): minimum point curve method

The equation (5) implies that T has a minimum value at

$$R = R_{\min} \equiv \sqrt{\frac{I_{CM}}{M+m}} \qquad \text{……………………. (11)}$$

Hence $R_{\min}$ can be obtained from the graph T v.s. R.

And therefore $$I_{CM} = (M+m) R_{\min}^2 \qquad \text{……………………. (12)}$$

This equation (12) together with equation (1) will allow us to calculate the required values z and $\frac{M}{m}$.

At the value $R = R_{\min}$ equation (5) becomes $T_{\min}^2 \frac{g(M+m)}{4\pi^2} = (M+m)R_{\min} + (M+m)R_{\min}$

$$g = \frac{2R_{\min}}{T_{\min}^2} \times 4\pi^2 = \frac{8\pi^2 R_{\min}}{T_{\min}^2} \qquad \text{……………………. (13)}$$

from which g can be calculated.

Results

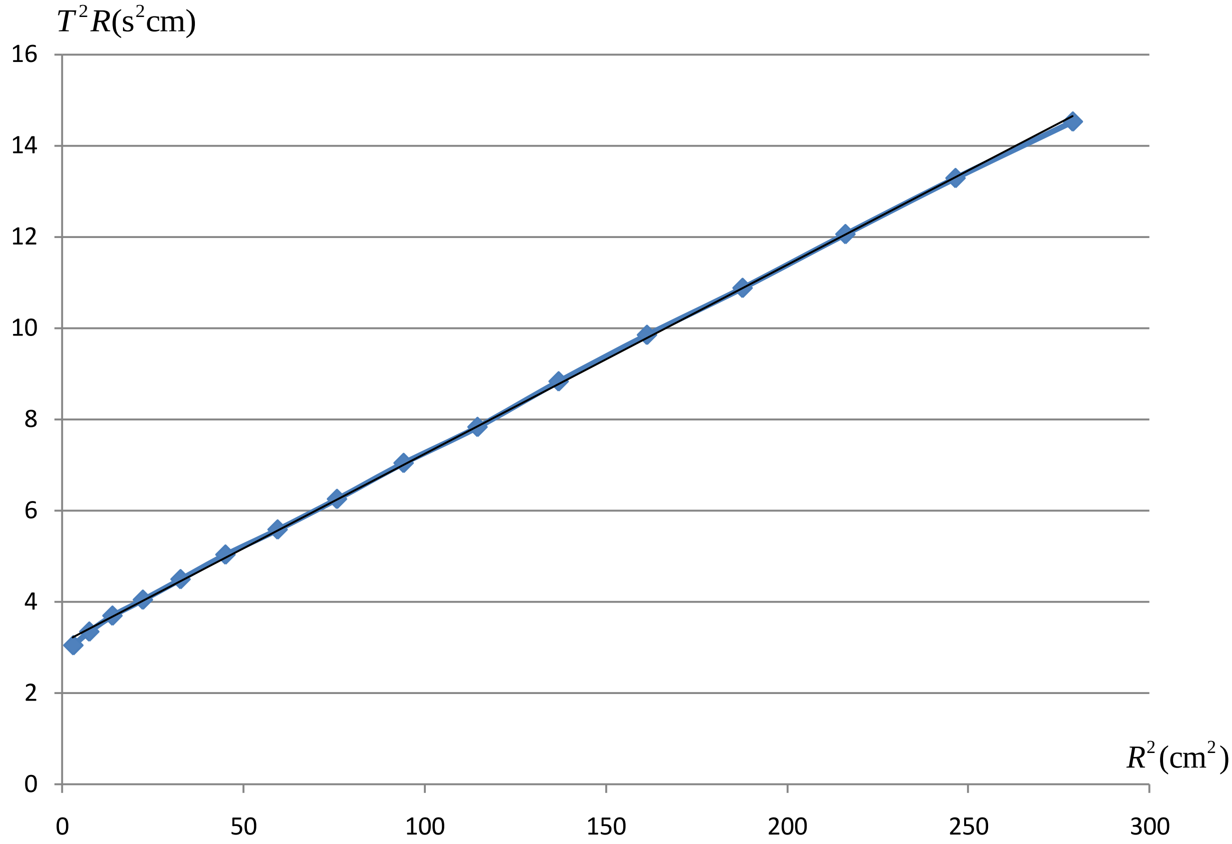
$$L = 30.0 \text{ cm} \pm 0.1 \text{ cm}$$

$$x_{CM} = 17.8 \text{ cm} \pm 0.1 \text{ cm (from top)}$$

$x_{CM} - R$ (cm)	time (s) for 20 cycles			T (s)	R (cm)	R^2 (cm^2)	T^2R (s^{2}cm)
1.1	18.59	18.78	18.59	0.933	16.7	278.9	14.53
2.1	18.44	18.25	18.53	0.920	15.7	246.5	13.29
3.1	18.10	18.09	18.15	0.906	14.7	216.1	12.06
4.1	17.88	17.78	17.81	0.891	13.7	187.7	10.88
5.1	17.69	17.50	17.65	0.881	12.7	161.3	9.85
6.1	17.47	17.38	17.28	0.869	11.7	136.9	8.83
7.1	17.06	17.06	17.22	0.856	10.7	114.5	7.83
8.1	17.06	17.00	17.06	0.852	9.7	94.1	7.04
9.1	16.97	16.91	16.96	0.847	8.7	75.7	6.25
10.1	17.00	17.03	17.06	0.852	7.7	59.3	5.58
11.1	17.22	17.37	17.38	0.866	6.7	44.9	5.03
12.1	17.78	17.72	17.75	0.888	5.7	32.5	4.49
13.1	18.57	18.59	18.47	0.927	4.7	22.1	4.04
14.1	19.78	19.90	19.75	0.991	3.7	13.7	3.69
15.1	11.16	11.13	11.13	1.114	2.7	7.3	3.34
16.1	13.25	13.40	13.50	1.338	1.7	2.9	3.04

Notes: at $x_{CM} - R = 15.1, 16.1$ cm, times for 10 cycles.

Method (a)



Calculation from straight line graph: slope $\alpha = 0.04108 \pm 0.0007$ s^2/cm, y-intercept $\beta = 3.10 \pm 0.05$ s^{2}cm

$$g = \frac{4\pi^2}{\alpha} \text{ giving } g = (961 \pm 20) \text{ cm/s}^2$$

$$\frac{\beta}{\alpha} = \frac{3.10}{0.04108} = 75.46 \text{ cm}^2 \left(\pm 2.5\,\text{cm}^2\right)$$

$$I_{CM} = (M+m)\frac{\beta}{\alpha} = (75.46)(M+m)$$

From equation (4): $I_{CM} = \frac{1}{3}M\left(\frac{L}{2}\right)^2 + M\left(x_{CM} - \frac{L}{2}\right)^2 + m\left(z - x_{CM}\right)^2$

Then $(75.46)(M+m) = 75.0M + 7.84M + m(z-17.8)^2$

$$-7.38\frac{M}{m} + 75.46 = (z-17.8)^2 \qquad \text{.........................} (14)$$

The centre of mass position gives:

$$17.8(M+m) = 15.0M + mz$$

$$\frac{M}{m} = \frac{z-17.8}{2.8} \qquad \text{.........................} (15)$$

From equations (14) and (15):

$$-\frac{7.38}{2.8}(z-17.8) + 75.46 = (z-17.8)^2$$

$$(z-17.8) = 7.47$$

And $z = 25.27 = 25.3 \pm 0.1 \text{ cm}$

$$\frac{M}{m} = 2.68 = 2.7$$

Error Estimation

Find error for g:

From (10), $g = \frac{4\pi^2}{\alpha}$

$$\Delta g = \frac{\Delta\alpha}{\alpha} g = 16.3 \text{ cm/s}^2 \approx 20 \text{ cm/s}^2$$

i) Find error for z:

First, find error for $r = \frac{\beta}{\alpha} = \frac{3.10}{0.04108} = 75.46 \text{ cm}^2$.

$$\Delta r = \left(\frac{\Delta\alpha}{\alpha} + \frac{\Delta\beta}{\beta}\right) r = 2.5 \text{ cm}^2$$

Since error from r contributes most ($\frac{\Delta r}{r} \sim 0.03$ while $\frac{\Delta L}{L}, \frac{\Delta x_{cm}}{x_{cm}} \sim 0.005$), we estimate error propagation from r only to simplify the analysis by substituting the min and max values into equation (4).

Now, we use $r_{max} = r + \Delta r = 75.46 + 2.5 = 77.96$. The corresponding quadratic equation is $(z-17.8)^2 + 1.743(z-17.8) - 77.96 = 0$ The corresponding solution is $(z-17.8)_{max} = 7.55$ cm

If we use $r_{\min} = r - \Delta r = 75.46 - 2.5 = 72.96$, the corresponding quadratic equation is

$$(z-17.8)^2 + 3.529(z-17.8) - 72.96 = 0$$

The corresponding solution is $(z-17.8)_{\min} = 6.96$ cm

So $\Delta(z-17.8) = \dfrac{7.55-6.96}{2} = 0.3$ cm

Note that $\dfrac{\Delta(z-17.8)}{z-17.8} \sim 0.04$. So, we still ignore the error propagation due to $\Delta L, \Delta x_{cm}$

The error Δz can be estimated from $\Delta z \approx \Delta(z-17.8) = 0.3$ cm

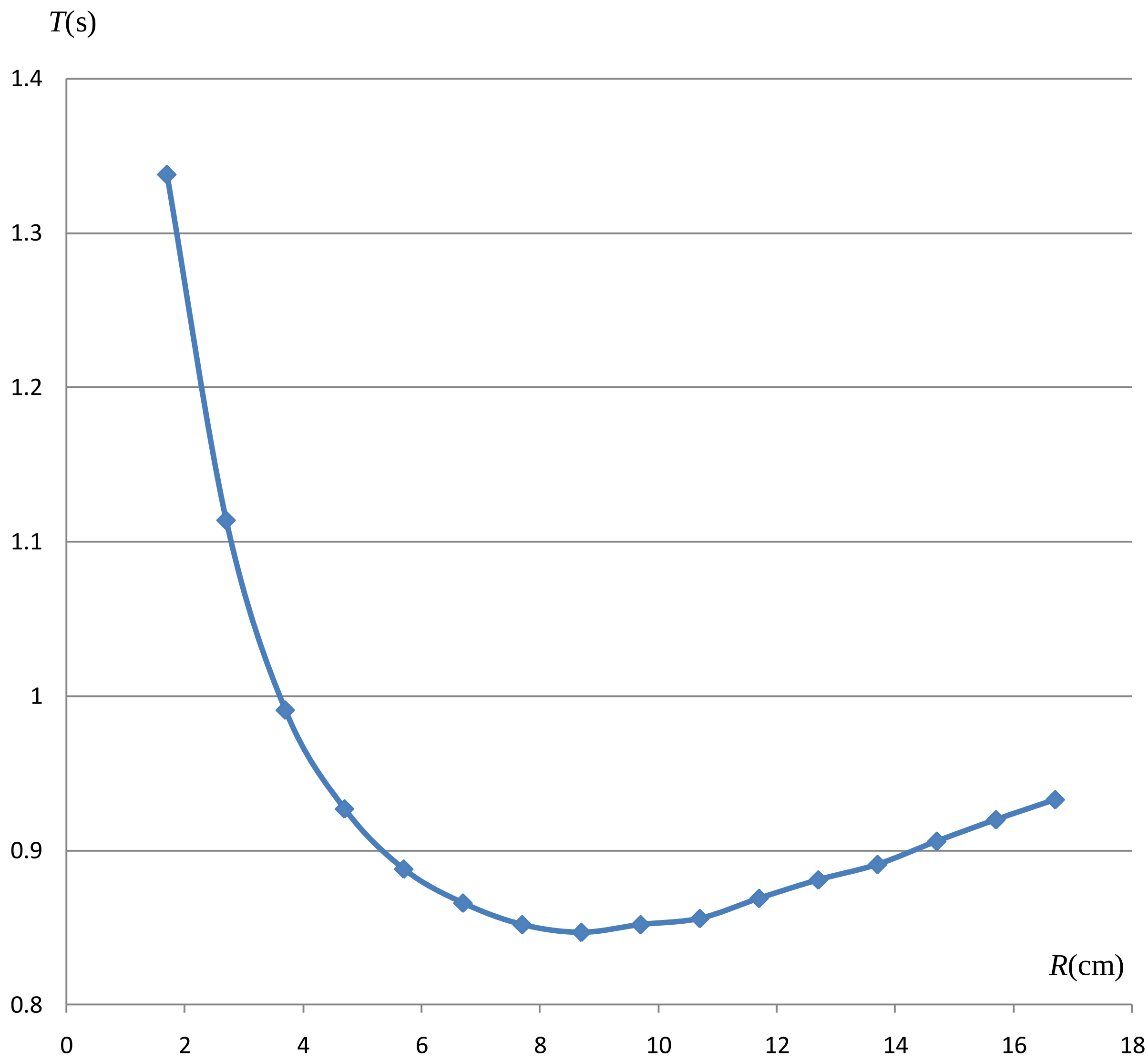
ii) Find error for $\dfrac{M}{m}$:

We know that $\dfrac{M}{m} = \dfrac{z-17.8}{2.8}$

$$\Delta\left(\frac{M}{m}\right) = \frac{\Delta(z-17.8)}{2.8} = 0.11$$

Method (b)

Calculation from *T*-*R* plot:



Using the minimum position: $T = T_{\min}$ at $I_{CM} = (M+m)R_{\min}^2$ and $g = \dfrac{8\pi^2 R_{\min}}{T_{\min}^2}$

From graph: $R_{\min} = 8.9 \pm 0.2$ cm and $T_{\min} = 0.846 \pm 0.005$ s

$$\therefore \quad g = 982 \pm 40 \text{ cm/s}^2$$

$$I_{CM} = (M+m)(8.9)^2 = (79.21)(M+m) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

From equations (14), (15), (16):

$$(79.21)(M+m) = 75.0M + 7.84M + m(z-17.8)^2$$

$$-3.63M + 79.21m = m(z-17.8)^2$$

$$(x-17.8)^2 + \frac{3.63}{2.8}(x-17.8) - 79.21 = 0$$

$$(z-17.8) = 8.28$$

And $z = 26.08 = 26.1 \pm 0.7$ cm

$$\frac{M}{m} = 2.95 = 3.0 \pm 0.3$$

Error estimation

i) Find error for g:

Using the minimum position: $g = \frac{8\pi^2 R_{\min}}{T_{\min}^2}$, we have

$$\Delta g = \left(\frac{\Delta R_{\min}}{R_{\min}} + 2\frac{\Delta T_{\min}}{T_{\min}}\right) g = 34 \approx 30 \text{ cm/s}^2$$

ii) Find error for z:

First, find error for $r = R_{\min}^2 = 79.21 \text{ cm}^2$.

$$\Delta r = 2R_{\min}\Delta R_{\min} = 3.56 \text{ cm}^2$$

This r is equivalent to r in part 1. So, one can follow the same error analysis.
As a result, we have

$$z = 26.08 \approx 26.1 \text{ cm}$$
$$\Delta z = 0.8 \text{ cm}$$

i) Find error for $\frac{M}{m}$:

Following the same analysis as in part I, we found that

$$\frac{M}{m} = 2.96 \,;\, \Delta(\frac{M}{m}) = 0.15$$

NOTE: This minimum curve method is not as accurate as the usual straight line graph.