A Festschrift in Honor of Gustavo Haddad Braga, the First Gold Medal for Brazil, Now the First Among the Ibero-American Countries in the History of the IPhOs to Receive Gold Medal - IPhO 42nd in Bangkok Thailand, 2011


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# International Physics Olympiads 1967-2011 <br> Part 2-XXV - XXXV - IPhO 1994-2004 

OMEGALEPH
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# International Physics Olympiads 1994-2004 

IPhO 1994-2004
Omegaleph Compilations

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# THE EXAMINATION <br> XXV INTERNATIONAL PHYSICS OLYMPIAD <br> BEIJING, PERPLE'S REPUBLIC CHINA <br> THEORETICAL COMPETITION <br> July 13, 1994 <br> Time available: 5 hours <br> READ THIS FIRST! 

## INSTRUCTIONS:

1. Use only the ball pen provided.
2. Your graphs should be drawn on the answer sheets attached to the problem.
3. Your solutions should be written on the sheets of paper attached to the problems.
4. Write at the top of the first page of each problem:

- The total number of pages in your solution to the problem
- Your name and code number


## Theoretical Problem 1

## RELATIVISTIC PARTICLE

In the theory of special relativity the relation between energy $E$ and momentum $P$ or a free particle with rest mass $m_{0}$ is

$$
E=\sqrt{p^{2} c^{2}+m_{0}^{2} c^{4}}=m c^{2}
$$

When such a particle is subject to a conservative force, the total energy of the particle, which is the sum of $\sqrt{p^{2} c^{2}+m_{0}^{2} c^{4}}$ and the potential energy, is conserved. If the energy of the particle is very high, the rest energy of the particle can be ignored (such a particle is called an ultra relativistic particle).

1) consider the one dimensional motion of a very high energy particle (in which rest energy can be neglected) subject to an attractive central force of constant magnitude $f$. Suppose the particle is located at the centre of force with initial momentum $p_{0}$ at time $t=0$. Describe the motion of the particle by separately plotting, for at least one period of the motion: $x$ against time $t$, and momentum $p$ against space coordinate $x$. Specify the coordinates of the "turning points" in terms of given parameters $p_{0}$ and $f$. Indicate, with arrows, the direction of the progress of the mothon in the ( $p, x$ ) diagram. There may be short intervals of time during which the particle is not ultrarelativistic. However, these should be neglected.
Use Answer Sheet 1.
2) A meson is a particle made up of two quarks. The rest mass $M$ of the meson is equal to the total energy of the two-quark system divided by $c^{2}$.

Consider a one--dimensional model for a meson at rest, in which the two quarks are assumed to move along the $x$-axis and attract each other with a force of constant magnitude f It is assumed they can pass through each other freely. For analysis of the high energy motion of the quarks the rest mass of the quarks can be neglected. At time $t=0$ the two quarks are both at $x=0$. Show separately the motion of the two quarks graphically by a ( $x, t$ ) diagram and a $(p, x)$ diagram, specify the coordinates of the "turning points" in terms of $M$ and $f$, indicate the direction of the process in your $(p, x)$ diagram, and determine the maximum distance between the two quarks.
Use Answer Sheet 2.
3) The reference frame used in part 2 will be referred to as frame $S$, the Lab frame, referred to as $S$, moves in the negative $x$-direction with a constant velocity $v=0.6 c$. the coordinates in the two reference frames are so chosen that the point
$x=0$ in $S$ coincides with the point $x^{\prime}=0$ in $S^{\prime \prime}$ at time $t=t^{\prime}=0$. Plot the motion of the two quarks graphically in a ( $x^{\prime}, t^{\prime}$ ) diagram. Specify the coordinates of the turning points in terms of $M, f$ and $c$, and determine the maximum distance between the two quarks observed in Lab frame $S^{\prime}$.

Use Answer Sheet 3.
The coordinates of particle observed in reference frames $S$ and $S^{\prime \prime}$ are related by the Lorentz transformation

$$
\left\{\begin{array}{l}
x^{\prime}=\gamma(x+\beta c t) \\
t^{\prime}=\gamma\left(t+\beta \frac{x}{c}\right)
\end{array}\right.
$$

where $\beta=v / c, \gamma=1 / \sqrt{1-\beta^{2}}$ and $v$ is the velocity of frame $S$ moving relative to the frame $S^{\prime \prime}$.
4) For a meson with rest energy $M c^{2}=140 \mathrm{MeV}$ and velocity $0.60 c$ relative to the Lab frame $S^{\prime \prime}$, determine its energy $E^{\prime}$ in the Lab Frame $S^{\prime \prime}$.

## ANSWER SHEET 1

1) 




## ANSWER SHEET 2

2) 




The maximum distance between the two quarks is $d=$

## ANSWER SHEET 3



## Theoretical Problem 1-Solution

1) 1a. Taking the force center as the origin of the space coordinate $x$ and the zero potential point, the potential energy of the particle is

$$
\begin{equation*}
U(x)=f|x| \tag{1}
\end{equation*}
$$

The total energy is

$$
W=\sqrt{p^{2} c^{2}+m_{0}^{2} c^{4}}+f|x| .
$$

1b. Neglecting the rest energy, we get

$$
\begin{equation*}
W=|p| c+f|x|, \tag{2}
\end{equation*}
$$

Since $W$ is conserved throughout the motion, so we have

$$
\begin{equation*}
W=|p| c+f|x|=p_{0} c, \tag{3}
\end{equation*}
$$

Let the $x$ axis be in the direction of the initial momentum of the particle,

$$
\left.\begin{array}{llll}
p c+f x=p_{0} c & \text { when } & x>0, & p>0 ; \\
-p c+f x=p_{0} c & \text { when } & x>0, & p<0 ; \\
p c-f x=p_{0} c & \text { when } & x<0, & p>0 ;  \tag{4}\\
-p c-f x=p_{0} c & \text { when } & x<0, & p<0 .
\end{array}\right\}
$$

The maximum distance of the particle from the origin, let it be $L$, corresponds to $p=0$. It is

$$
L=p_{0} c / f .
$$

1c. From Eq. 3 and Newton's law

$$
\frac{d p}{d t}=F= \begin{cases}-f, & x>0  \tag{5}\\ f, & x<0\end{cases}
$$

we can get the speed of the particle as

$$
\begin{equation*}
\left|\frac{d x}{d t}\right|=\frac{c}{f}\left|\frac{d p}{d t}\right|=c, \tag{6}
\end{equation*}
$$

i.e. the particle with very high energy always moves with the speed of light except that it is in the region extremely close to the points $x= \pm L$. The time for the particle to move from origin to the point $x=L$, let it be denoted by $\tau$, is

$$
\tau=L / c=p_{0} / f .
$$

So the particle moves to and for between $x=L$ and $x=-L$ with speed $c$ and period $4 \tau=4 p_{0} / f$. The relation between $x$ and $t$ is

$$
\left.\begin{array}{ll}
x=c t, & 0 \leq t \leq \tau \\
x=2 L-c t, & \tau \leq t \leq 2 \tau,  \tag{7}\\
x=2 L-c t, & 2 \tau \leq t \leq 3 \tau, \\
x=c t-4 L, & 3 \tau \leq t \leq 4 \tau,
\end{array}\right\}
$$

The required answer is thus as given in Fig. 1 and Fig. 2.


Fig. 1


Fig. 2
2) The total energy of the two-quark system can be expressed as

$$
\begin{equation*}
M c^{2}=\left|p_{1}\right| c+\left|p_{2}\right| c+f\left|x_{1}-x_{2}\right|, \tag{8}
\end{equation*}
$$

where $x_{1}, x_{2}$ are the position coordinates and $p_{1}, p_{2}$ are the momenta of quark 1 and quark 2 respectively. For the rest meson, the total momentum of the two quarks is zero and the two quarks move symmetrically in opposite directions, we have

$$
\begin{equation*}
p=p_{1}+p_{2}=0, \quad p_{1}=-p_{2}, \quad x_{1}=-x_{2} . \tag{9}
\end{equation*}
$$

Let $p_{0}$ denote the momentum of the quark 1 when it is at $x=0$, then we have

$$
\begin{equation*}
M c^{2}=2 p_{0} c \quad \text { or } \quad p_{0}=M c / 2 \tag{10}
\end{equation*}
$$

From Eq. 8, 9 and 10, the half of the total energy can be expressed in terms of $p_{1}$ and $x_{1}$ of quark 1:

$$
\begin{equation*}
p_{0} c=\left|p_{1}\right| c+f\left|x_{1}\right|, \tag{11}
\end{equation*}
$$

just as though it is a one particle problem as in part 1 (Eq. 3) with initial momentum
$p_{0}=M c / 2$. From the answer in part 1 we get the $(x, t)$ diagram and $(p, x)$ diagram of the motion of quark 1 as shown in Figs. 3 and 4. For quark 2 the situation is similar except that the signs are reversed for both $x$ and $p$; its ( $x, t$ ) and ( $p, x$ ) diagrams are shown in Figs. 3 and 4.

The maximum distance between the two quarks as seen from Fig. 3 is

$$
\begin{equation*}
d=2 L=2 p_{0} c / f=M c^{2} / f . \tag{12}
\end{equation*}
$$



Fig. 3


Fig. 4a
Fig. 4b

Quark1
Quark2
3) The reference frame $S$ moves with a constant velocity $V=0.6 c$ relative to the Lab frame $S^{\prime \prime}$ in the $x^{\prime}$ axis direction, and the origins of the two frames are coincident at the beginning $\left(t=t^{\prime}=0\right)$. The Lorentz transformation between these two frames is given by:

$$
\begin{align*}
& x^{\prime}=\gamma(x+\beta c t),  \tag{13}\\
& t^{\prime}=\gamma(t+\beta x / c),
\end{align*}
$$

where $\beta=V / c$, and $\gamma=1 / \sqrt{1-\beta^{2}}$. With $V=0.6 c$, we have $\beta=3 / 5$, and $\gamma=5 / 4$. Since the Lorenta transformation is linear, a straight line in the $(x, t)$ diagram
transforms into a straight line the ( $x^{\prime}, t^{\prime}$ ) diagram, thus we need only to calculate the coordinates of the turning points in the frame $S^{\prime}$.

For quark 1, the coordinates of the turning points in the frames $S$ and $S^{\prime}$ are as follows:

Frame $S$
$X_{1} \quad t_{1}$

$$
\begin{aligned}
& x_{1}^{\prime}=\gamma\left(x_{1}+\beta c t_{1}\right) \\
= & \frac{5}{4} x_{1}+\frac{3}{4} c t_{1}
\end{aligned}
$$

$0 \quad 0$
$L \quad \tau$
$\tau \quad \gamma(1+\beta) L=2 L$
$2 \gamma \beta L=\frac{3}{2} L$
$\gamma(3 \beta-1) L=L$
$\tau \quad 4 \gamma \beta L=3 L$
$0 \quad 2 \tau$
$-L \quad 3 \tau$
$0 \quad 4 \tau$

Frame $S^{\prime}$

$$
\begin{aligned}
& t_{1}^{\prime}=\gamma\left(t_{1}+\beta x_{1} / e\right) \\
= & \frac{5}{4} t_{1}+\frac{3}{4} x_{1} / c
\end{aligned}
$$

0
$2 \gamma \tau=\frac{5}{2} \tau$
$\gamma(3-\beta) \tau=3 \tau$
$4 \gamma \tau=5 \tau$
where $L=p_{0} c / f=M c^{2} / 2 f, \tau=p_{0} / f=M c / 2 f$.
For quark 2, we have
Frame $S$
Frame $S^{\prime}$

$$
\begin{array}{lll}
x_{2} \quad t_{2} & x_{2}^{\prime}=\gamma\left(x_{2}+\beta c t_{2}\right) \\
& =\frac{5}{4} x_{2}+\frac{3}{4} c t_{2}
\end{array}
$$

$0 \quad 0$
0
$-\gamma(1-\beta) L=-\frac{1}{2} L$
$\gamma(1-\beta) \tau=\frac{1}{2} \tau$
$0 \quad 2 \tau$
$2 \gamma \beta L=\frac{3}{2} L$
$2 \gamma \tau=\frac{5}{2} \tau$
$L \quad 3 \tau$
$\gamma(3 \beta+1) L=\frac{7}{2} L$
$\gamma(3+\beta) \tau=\frac{9}{2} \tau$
$0 \quad 4 \tau$
$4 \gamma \beta L=3 L$
$4 \gamma \tau=5 \tau$
With the above results, the ( $x^{\prime}, t^{\prime}$ ) diagrams of the two quarks are shown in Fig. 5.
The equations of the straight lines $O A$ and $O B$ are:

$$
\begin{array}{ll}
x_{1}^{\prime}\left(t^{\prime}\right)=c t^{\prime} ; & 0 \leq t^{\prime} \leq \gamma(1+\beta) \tau=2 \tau ; \\
x_{2}^{\prime}\left(t^{\prime}\right)=-c t^{\prime} ; & 0 \leq t^{\prime} \leq \gamma(1-\beta) \tau=\frac{1}{2} \tau \tag{14b}
\end{array}
$$

The distance between the two quarks attains its maximum $d^{\prime}$ when $t^{\prime}=\frac{1}{2} \tau$, thus we have maximum distance

$$
\begin{equation*}
d^{\prime}=2 c \gamma(1-\beta) \tau=2 \gamma(1-\beta) L=\frac{M c^{2}}{2 f} . \tag{15}
\end{equation*}
$$



Fig. 5
4) It is given the meson moves with velocity $V=0.6$ crelative to the Lab frame, its energy measured in the Lab frame is

$$
E^{\prime}=\frac{M c^{2}}{\sqrt{1-\beta^{2}}}=\frac{1}{0.8} \times 140=175 \mathrm{MeV}
$$

## Grading Scheme

Part 12 points, distributed as follows:
0.4 point for the shape of $x(t)$ in Fig. 1;
0.3 point for 4 equal intervals in Fig. 1;
( 0.3 for correct derivation of the formula only)
0.1 each for the coordinates of the turning points $A$ and $C, 0.4$ point in total;
0.4 point for the shape of $p(x)$ in fig. 2; ( 0.2 for correct derivation only)
0.1 each for specification of $p_{0}, L=p_{0} c / f,-p_{0},-L$ and arrows, 0.5 point in total.
( 0.05 each for correct calculations of coordinate of turning points only).
Part 24 points, distributed as follows:
0.6 each for the shape of $x_{1}(t)$ and $x_{2}(t), 1.2$ points in total;
0.1 each for the coordinates of the turning points A, B, D and E in Fig. 3, 0.8 point in total;
0.3 each for the shape of $p_{1}\left(x_{1}\right)$ and $p_{2}\left(x_{2}\right), 0.6$ point in total;
0.1 each for $p_{0}=M c / 2, L=M c^{2} / 2 f,-p_{0},-L$ and arrows in Fig. 4a and Fig. 4b, 1 point in total;
0.4 point for $d=M c^{2} / f$

Part 33 point, distributed as follows:
0.8 each for the shape of $x_{1}^{\prime}\left(t^{\prime}\right)$ and $x_{2}^{\prime}\left(t^{\prime}\right), 1.6$ points in total;
0.1 each for the coordinates of the turning points $\mathrm{A}, \mathrm{B}, \mathrm{D}$ and E in Fig. 5, 0.8 point in total; ( 0.05 each for correct calculations of coordinate of turning points only).
0.6 point for $d^{\prime}=M c^{2} / 2 f$.

Part 41 point ( 0.5 point for the calculation formula; 0.5 point for the numerical value and units)

## Theoretical Problem 2

## SUPERCONDUCTING MAGNET

Super conducting magnets are widely used in laboratories. The most common form of super conducting magnets is a solenoid made of super conducting wire. The wonderful thing about a superconducting magnet is that it produces high magnetic fields without any energy dissipation due to Joule heating, since the electrical resistance of the superconducting wire becomes zero when the magnet is immersed in liquid helium at a temperature of 4.2 K . Usually, the magnet is provided with a specially designed superconducting switch, as shown in Fig. 1. The resistance $r$ of the switch can be controlled: either $r=0$ in the superconducting state, or $r=r_{n}$ in the normal state. When the persistent mode, with a current circulating through the magnet and superconducting switch indefinitely. The persistent mode allows a steady magnetic field to be maintained for long periods with the external source cut off.

The details of the superconducting switch are not given in Fig. 1. It is usually a small length of superconducting wire wrapped with a heater wire and suitably thermally insulated from the liquid helium bath. On being heated, the temperature of the superconducting wire increases and it reverts to the resistive normal state. The typical value of $r_{n}$ is a few ohms. Here we assume it to be $5 \Omega$. The inductance of a superconducting magnet depends on its size; assume it be 10 H for the magnet in Fig. 1. The total current $I$ can be changed by adjusting the resistance $R$.

This problem will be graded by the plots only!
The arrows denote the positive direction of $I, I_{1}$ and $I_{2}$.


Fig. 1

1) If the total current $I$ and the resistance $r$ of the superconducting switch are controlled
to vary with time in the way shown in Figs, 2a and 2 b respectively, and assuming the currents $I_{1}$ and $I_{2}$ flowing through the magnet and the switch respectively are equal at the beginning (Fig. 2c and Fig. 2d), how do they vary with time from $t_{1}$ to $t_{4}$ ? Plot your answer in Fig. 2c and Fig. 2d

2) Suppose the power switch $K$ is turned on at time $t=0$ when $r=0, I_{1}=0$ and $R=7.5 \Omega$, and the total current $I$ is 0.5 A . With $K$ kept closed, the resistance $r$ of the superconducting switch is varied in he way shown in Fig. 3b. Plot the corresponding time dependences of $I, I_{1}$ and $I_{2}$ in Figs. 3a, 3c and 3d respectively.

3) Only small currents, less than 0.5 A , are allowed to flow through the
superconducting switch when it is in the normal state, with larger currents the switch will be burnt out. Suppose the superconducting magnet is operated in a persistent mode, i. e. $I=0$, and $I_{1}=i_{1}$ (e. g. 20A), $I_{2}=-i_{1}$, as shown in Fig. 4, from $t=0$ to $t=3 \mathrm{~min}$. If the experiment is to be stopped by reducting the current through the magnet to zero, how would you do it? This has to be done in several operation steps. Plot the corresponding changes of $I, r, I_{1}$ and $I_{2}$ in Fig. 4


4b


4d
4) Suppose the magnet is operated in a persistent mode with a persistent current of 20 A ( $t=0$ to $t=3 \mathrm{~min}$. See Fig. 5). How would you change it to a persistent mode with a current of 30a? plot your answer in Fig. 5.


Fig. 5a


5c



## Theoretical Problem 2-Solution

1) For $t=t_{1}$ to $t_{3}$

Since $r=0$, the voltage across the magnet $V_{M}=L d I_{1} / d t=0$, therefore,

$$
\begin{gathered}
I_{1}=I_{1}\left(t_{1}\right)=\frac{1}{2} I_{0} ; \\
I_{2}=I-I_{1}=I-\frac{1}{2} I_{0} .
\end{gathered}
$$

For $t=t_{3}$ to $t_{4}$
Since $I_{2}=0$ at $t=t_{3}$, and $I$ is kept at $\frac{1}{2} I_{0}$ after
$t=t_{3}, V_{M}=I_{2} r_{n}=0$, therefore, $I_{1}$ and $I_{2}$ will not change.

$$
\begin{gathered}
I_{1}=\frac{1}{2} I_{0} ; \\
I_{2}=0
\end{gathered}
$$

These results are shown in Fig. 6.


Fig. 6a

6c

6d
2) For $t=0$ to $t=1 \mathrm{~min}$ :

Since $r=0, V_{M}=L d I_{1} / d t=0$

$$
\begin{aligned}
& I_{1}=I_{1}(0)=0 \\
& I_{2}=I-I_{1}=0.5 \mathrm{~A} .
\end{aligned}
$$

At $t=1$ min, $r$ suddenly jumps from O to $r_{n}$, I will drop from $E / R$ to $E /\left(R+r_{n}\right)$ instantaneously, because $I_{1}$ can not change abruptly due to the inductance of the magnet coil. For $E / R=0.5 \mathrm{~A}, R=7.5 \Omega$ and $R_{n}=5 \Omega$. I will drop to 0.3 A .

For $t=1 \mathrm{~min}$ to 2 min :
$I, I_{1}$ and $I_{2}$ gradually approach their steady values:

$$
\begin{aligned}
& I=\frac{E}{R}=0.5 \mathrm{~A}, \\
& I_{1}=I=0.5 \mathrm{~A} \\
& I_{2}=0 .
\end{aligned}
$$

The time constant

$$
\tau=\frac{L\left(R+r_{n}\right)}{R r_{n}} .
$$

When $L=10 \mathrm{H}, R=7.5 \Omega$ and $r_{n}=5 \Omega, \tau=3 \mathrm{sec}$.
For $t=2 \mathrm{~min}$ to 3 min :
Since $r=0, I_{1}$ and $I_{2}$ will not change, that is

$$
I_{1}=0.5 \mathrm{~A} \text { and } I_{2}=0
$$



Fig. 7a


3) The operation steps are:

## First step

Turn on power switch $K$, and increase the total current $I$ to 20 A , i. e. equal to $I_{1}$. Since the superconducting switch is in the state $r=0$, so that $V_{M}=L d I_{1} / d t=0$, that is, $I_{1}$ can not change and $I_{2}$ increases by 20 A , in other words, $I_{2}$ changes from - 20 A to zero.

## Second step

Switch $r$ from 0 to $r_{n}$.

## Third step

Gradually reduce $I$ to zero while keeping $I_{2}<0.5 \mathrm{~A}$ : since $I_{2}=V_{M} / r_{n}$ and $V_{m}=L d I_{1} / d t$, when $L=10 \mathrm{H}, r_{n}=5 \Omega$, the requirement $I_{2}<0.5 \mathrm{~A}$ corresponds to $d I_{1} / d t<0.25 \mathrm{~A} / \mathrm{sec}$, that is, a drop of $<15 \mathrm{~A}$ in 1 min . In Fig. $8 d I / d t \sim 0.1 \mathrm{~A} / \mathrm{sec}$ and $d I_{1} / d t$ is around this value too, so the requirement has been fulfilled.

## Final step

Switch $r$ to zero when $V_{M}=0$ and turn off the power switch $K$. These results are shown in Fig. 8.


Fig. 8a


8b

8c


4) First step and second step are the same as that in part 3, resulting in $I_{2}=0$.

Third step Increase I by 10 A to 30 A with a rate subject to the requirement $I_{2}<0.5 \mathrm{~A}$.

Fourth step Switch $r$ to zero when $V_{M}=0$.

Fifth step Reduce $I$ to zero, $I_{1}=30$ A will not change because $V_{M}$ is zero. $I_{2}=I-I_{1}$ will change to -30 A . The current flowing through the magnet is thus closed by the superconducting switch.

Final step Turn off the power switch $K$. The magnet is operating in the persistent mode.

These results are shown in Fig. 9.


Fig. 9a

9b


9c


## Grading Scheme

Part 1, 2 points:
0.5 point for each of $I_{1}, I_{2}$ from $t=t_{1}$ to $t_{3}$ and $I_{1}, I_{2}$ from $t=t_{3}$ to $t_{4}$. Part 2, $\quad 3$ points:
0.3 point for each of $I_{1}, I_{2}$ from $t=0$ to $1 \mathrm{~min}, I, I_{1}, I_{2}$ at $t=1 \mathrm{~min}$,
and $I_{0}, I_{1}, I_{2}$ from $t=1$ to 2 min ;
0.2 point for each of $I, I_{1}$, and $I_{2}$ from $t=2$ to 3 min.

Part 3, 2 points:
0.25 point for each section in Fig. 8 from $t=3$ to 9 min , 8 sections in total.

Part 4, 3 points:
0.25 point for each section in Fig. 9 from $t=3$ to $12 \mathrm{~min}, 12$ sections in total.

## Theoretical Problem 3

## COLLISION OF DISCS WITH SURFACE FRICTION

A homogeneous disc A of mass m and radius $R_{A}$ moves translationally on a smooth horizontal $x-y$ plane in the $x$ direction with a velocity $V$ (see the figure on the next page). The center of the disk is at a distance $b$ from the x -axis. It collides with a stationary homogeneous disc B whose center is initially located at the origin of the coordinate system. The disc B has the same mass and the same thickness as A, but its radius is $R_{B}$. It is assumed that the velocities of the discs at their point of contact, in the direction perpendicular to the line joining their centers, are equal after the collision. It is also assumed that the magnitudes of the relative velocities of the discs along the line joining their centers are the same before and after the collision.

1) For such a collision determine the $X$ and $Y$ components of the velocities of the two discs after the collision, i. e. $V_{A X}^{\prime}, V_{A Y}^{\prime}, V_{B X}^{\prime}$ and $V_{B Y}^{\prime}$ in terms of $m, R_{A}, R_{B}$, $V$ and $b$.
2) Determine the kinetic energies $E_{A}^{\prime}$ for disc $A$ and $E_{B}^{\prime}$ for disc $B$ after the collision in terms of $m, R_{A}, R_{B}, V$ and $b$.


## Theoretical Problem 3-Solution

1) When disc A collides with disc $B$, let $n$ be the unit vector along the normal to the surfaces at the point of contact and $t$ be the tangential unit vector as shown in the figure. Let $\varphi$ be the angle between $n$ and the $x$ axis. Then we have

$$
b=\left(R_{A}+R_{B}\right) \sin \varphi
$$

The momentum components of $A$ and $B$ along $n$ and $t$ before collision are:

$$
m V_{A n}=m V \cos \varphi, m V_{B n}=0,
$$

$$
m V_{A t}=m V \sin \varphi, m V_{B t}=0 .
$$

Denote the corresponding momentum components of $A$ and $B$ after collision by $m V_{A n}^{\prime}, m V_{B n}^{\prime}, m V_{A t}^{\prime}$, and $m V_{B t}^{\prime}$. Let $\omega_{A}$ and $\omega_{B}$ be the angular velocities of $A$ and $B$ about the axes through their centers after collision, and $I_{A}$ and $I_{B}$ be their corresponding moments of intertia. Then,

$$
I_{A}=\frac{1}{2} m R_{A}^{2}, \quad I_{B}=\frac{1}{2} m R_{B}^{2}
$$

The conservation of momentum gives

$$
\begin{align*}
& m V \cos \varphi=m V_{A n}^{\prime}+m V_{B n}^{\prime},  \tag{1}\\
& m V \sin \varphi=m V_{A t}^{\prime}+m V_{t n}^{\prime}, \tag{2}
\end{align*}
$$

The conservation of angular momentum about the axis through O gives

$$
\begin{equation*}
m V b=m V_{A t}^{\prime}\left(R_{A}+R_{B}\right)+I_{A} \omega_{A}+I_{B} \omega_{B} \tag{3}
\end{equation*}
$$

The impulse of the friction force exerted on B during collision will cause a momentum change of $m V_{A t}^{\prime}$ along $t$ and produces an angular momentum $I_{B} \omega_{B}$ simultaneously. They are related by.

$$
\begin{equation*}
m V_{B t}^{\prime} R_{b}=I_{B} \omega_{B} \tag{4}
\end{equation*}
$$



During the collision at the point of contact A and B acquires the same tangential velocities, so we have

$$
\begin{equation*}
V_{A t}^{\prime}-\omega_{A} R_{A}=V_{B t}^{\prime}-\omega_{B} R_{B} \tag{5}
\end{equation*}
$$

It is given that the magnitudes of the relative velocities along the normal direction of the two discs before and after collision are equal, i. e.

$$
\begin{equation*}
V \cos \varphi=V_{B n}^{\prime}-V_{A n}^{\prime} . \tag{6}
\end{equation*}
$$

From Eqs. 1 and 6 we get

$$
\begin{aligned}
& V_{A n}^{\prime}=0, \\
& V_{B n}^{\prime}=V \cos \varphi .
\end{aligned}
$$

From Eqs. 2 to 5, we get

$$
\begin{aligned}
V_{A t}^{\prime} & =\frac{5}{6} V \sin \varphi, \\
V_{B t}^{\prime} & =\frac{1}{6} V \sin \varphi, \\
\omega_{A} & =\frac{V \sin \varphi}{3 R_{A}}, \\
\omega_{B} & =\frac{V \sin \varphi}{3 R_{B}} .
\end{aligned}
$$

The $x$ and $y$ components of the velocities after collision are:

$$
\begin{align*}
& V_{A x}^{\prime}=V_{A n}^{\prime} \cos \varphi+V_{A t}^{\prime} \sin \varphi=\frac{5 V b^{2}}{6\left(R_{A}+R_{B}\right)^{2}},  \tag{7}\\
& V_{A y}^{\prime}=-V_{A n}^{\prime} \sin \varphi+V_{A t}^{\prime} \cos \varphi=\frac{5 V b \sqrt{\left(R_{A}+R_{B}\right)^{2}-b^{2}}}{6\left(R_{A}+R_{B}\right)^{2}},  \tag{8}\\
& V_{B x}^{\prime}=V_{B n}^{\prime} \cos \varphi+V_{B t}^{\prime} \sin \varphi=\left[1-\frac{5 b^{2}}{6\left(R_{A}+R_{B}\right)^{2}}\right],  \tag{9}\\
& V_{B y}^{\prime}=-V_{B n}^{\prime} \sin \varphi+V_{B t}^{\prime} \cos \varphi=-\frac{5 V b \sqrt{\left(R_{A}+R_{B}\right)^{2}-b^{2}}}{6\left(R_{A}+R_{B}\right)^{2}}, \tag{10}
\end{align*}
$$

2) After the collision, the kinetic energy of disc $A$ is

$$
\begin{equation*}
E_{A}^{\prime}=\frac{1}{2} m\left(V_{A x}^{\prime 2}+V_{A y}^{\prime 2}\right)+\frac{1}{2} I_{A} \omega_{A}^{2}=\frac{3 m V^{2} b^{2}}{8\left(R_{A}+R_{B}\right)^{2}} \tag{11}
\end{equation*}
$$

while the kinetic energy of disc $B$ is

$$
\begin{equation*}
E_{B}^{\prime}=\frac{1}{2} m\left(V_{B x}^{\prime 2}+V_{B y}^{\prime 2}\right)+\frac{1}{2} I_{B} \omega_{B}^{2}=\frac{1}{2} m V^{2}\left[1-\frac{11 b^{2}}{12\left(R_{A}+R_{B}\right)^{2}}\right] \tag{12}
\end{equation*}
$$

## Grading Scheme

1. After the collision, the velocity components of discs A and B are shown in Eq. 7, 8,9 and 10 of the solution respectively. The total points of this part is 8 . 0 . If the result in which all four velocity components are correct has not been obtained, the point is marked according to the following rules.
0.8 point for each correct velocity component;
0.8 point for the correct description of that the magnitudes of the relative velocities of the discs along the line joining their centers are the same before and after the collision.
0.8 point for the correct description of the conservation for angular momentum;
0.8 point for the correct description of the equal tangential velocity at the touching point;
0.8 point for the correct description of the relation between the impulse and the moment of the impulse.
2. After the collision, the kinetic energies of disc A and disc B are shown in Eqs. 11 and 12 of the solution respectively.
1.0 point for the correct kinetic energies of disc A ;
1.0 point for the correct kinetic energies of disc $B$;

The total points of this part is 2.0

# XXV INTERNATIONAL PHYSICS OLYMPIAD BEIJING, P EOPLE'S REPUBLIC OF CHINA PRACTICAL COMPETITION 

July 15, 1994

Time available: 2.5 hours

## READ THIS FIRST!

## INSTRUCTIONS:

1. Use only the ball pen provided.
2. Your graphs should be drawn on the answer sheets attached to the problem.
3. Write your solution on the marked side of the paper only.
4. The draft papers are provided for doing numerical calculations and draft drawings.
5. Write at the top of every page:

- The number of the problem
- The number of the page of your report in each problem
- The total number of pages in your report to the problem
- Your name and code number


## EXPERIMENTAL PROBLEM 1

Determination of light reflectivity of a transparent dielectric surface.

## Experimental Apparatus

1. He-Ne Laser $(\sim 1.5 \mathrm{~mW})$.The light from this laser is not linearly polarized.
2. Two polarizers ( $\mathbf{P}_{1}, \mathbf{P}_{2}$ ) with degree scale disk (Fig. 1), one ( $\mathbf{P}_{1}$ ) has been mounted in front of the laser output window as a polarizer, and another one can be fixed in a proper place of the drawing board by push-pins when it is necessary.
3. Two light intensity detectors ( $\mathrm{D}_{1}, \mathrm{D}_{2}$ ) which consisted of a photocell and a microammeter (Fig. 2).
4. Glass beam splitter(B).
5. Transparent dielectric plate, whose reflectivity and refractive index are to be determined.
6. Sample table mounted on a semicircular degree scale plate with a coaxial swivel arm(Fig. 3).
7. Several push-.pins for fixing the sample table on the drawing board and as its rotation axis.
8. Slit aperture and viewing screen for adjusting the laser beam in the horizontal direction and for alignment of optical elements.
9. Lute for adhere of optical elements in a fixed place.
10. Wooden drawing board.
11. Plotting papers

## Experiment Requirement

1. Determine the reflectivity of the p-component as a function of the incident angle (the electric field component, parallel to the plane of incidence is called the $p$-component).
(a) Specify the transmission axis of the polarizer (A) by the position of the marked line on the degree scale disk in the $p$-componet measurement(the transmission axis is the direction of vibration of the electric field vector of the transmitted light).
(b) Choose any one of the light intensity detector and set its micro-ammeter at the range of " $\times 5$ ". Verify the linear relation ship between the light intensity and the micro-ammeter reading. Draw the optical schematic diagram. Show your measured data and calculated results(including the calculation formula)in the farm of a table. Plot the linear relationship curve.
(c) Determine the reflectivity of the $p$-component as a function of the incident angle. Draw the optical schematic diagram. Show your measured data and calculated reflectivity(including the calculation formula)in the form of a table. Plot the reflectivity as a function of the incident angle.
2. Determine the refractive index of the sample as accurate as possible.

## Explanation and Suggestion

1. Laser radiation avoid direct eye exposure.
2. Since the output power of the laser beam may fluctuate from time to time, the fluctuation of light output has to be monitored during the performance of the experiment and a correction of the experimental results has to be made.
3. The laser should be lighting all the time, even when you finish your experiment and leave the examination hall, the laser should be keeping in work.
4. The reflected light is totally plane polarized at an incident angle $\theta_{B}$ while $\operatorname{tg} \theta_{B}=n \quad$ (refractive index).


Fig. 1 polarizers with degree scale disk


Fig. 2 Light intensity detector
(1) Insert the plug of photocell into the "INPUT" socket of microammeter
(2) Switching on the microammeter.
(3) Blocd off the light entrance hole in front of the photocell and adjust the scale reading of micro ammeter to " 0 ".
(4) Set the "Multiple" knob to a proper range.


Fig. 3 Sample table mounted on a semicircular degree scale plate

## Experimental Problem 1-Solution

1. (a) Determine the transmission axis of the polarizer and the Brewster angle $\theta_{B}$ of the sample by using the fact that the rerlectivity of the $p$-component $R_{p}=0$ at the Brewster angle.

Change the orientation of the transmission axis of $P_{1}$, specified by the position of the marked line on the degree scale disk $(\psi)$ and the incident angle $\left(\theta_{i}\right)$ successively until the related intensity $I_{r}=0$.


Now the incident light consists of $p$-component only and the incident angle is $\theta_{B}$, the
corresponding values $\psi_{1}$ and $\theta_{B}$ are shown below:

| $\psi_{1}$ | $140.0^{\circ}$ | $322.0^{\circ}$ | $141.0^{\circ}$ | $322.5^{\circ}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\theta$ | $56.4^{\circ}$ | $56.4^{\circ}$ | $56.2^{\circ}$ | $56.2^{\circ}$ |

$\psi_{1}=140.5^{\circ} \pm 0.5^{\circ}$ or $322.3^{\circ} \pm 0.1^{\circ}$
The Brewster angle $\theta_{B}$ is $56.3^{\circ} \pm 0.1^{\circ}$

1. (b) Verification of the linear relationship between the light intensity and the microammenter reading.


The intensity the transmitted light passing through two polarized $P_{1}$ and $P_{2}$ obeys Malus' law

$$
I(\theta)=I_{0} \cos ^{2} \theta
$$

where $I_{0}$ is the intensity of the light polarized by $p_{1}$ and incident, $I$ is the intensity of the transmitted light, and $\theta$ is the angle between the transmission axes of $P_{1}$ and $p_{2}$. Thus we can obtain light with various intensities for the verification by using two polarizers.

The experimental arrangement is shown in the figure.
The light intensity detector $D_{1}$ serves to monitor the intensity fluctuation of the incident beam (the ratio of $I_{1}$ to $I_{2}$ remain unchanged), and $D_{2}$ measures $I_{2}$. Let $i_{1}(\theta)$ and $i_{2}(\theta)$ be the readings of $D_{1}$ and $D_{2}$ respectively, and $\psi_{2}(\theta)$ be the reading of the marked line position. $i_{2}=0$ when $\theta=90^{\circ}$, the corresponding $\psi_{2}$ is $\psi_{2}\left(90^{\circ}\right)$, and the value of $\theta$ corresponding to $\psi_{2}$ is

$$
\theta=\left|\psi_{2}-\psi_{2}\left(90^{\circ}\right) \pm 90^{\circ}\right|
$$

Data and results;

$$
\psi_{2}\left(90^{\circ}\right)=4^{\circ}
$$

| $\psi_{2}$ | $94.0^{\circ}$ | $64.0^{\circ}$ | $49.0^{\circ}$ | $34.0^{\circ}$ | $4.0^{\circ}$ |
| :---: | :---: | :--- | :--- | :--- | :--- |
| $\theta$ | $0.0^{\circ}$ | $30.0^{\circ}$ | $45.0^{\circ}$ | $60.0^{\circ}$ | $90.0^{\circ}$ |
| $i_{1}(\theta) \mu A$ | $6.3 \times 1$ | $5.7 \times 1$ | $5.7 \times 1$ | $5.7 \times 1$ | $5.7 \times 1$ |
| $i_{2}(\theta) \mu A$ | $18.7 \times 5$ | $12.7 \times 5$ | $8.2 \times 5$ | $4.0 \times 5$ | $0.0 \times 5$ |

From the above data we can obtain the values of $I(\theta) / I_{2}(\theta)$ from the formula

$$
\frac{I(\theta)}{I_{0}}=\frac{i_{2}(\theta)}{i_{1}(\theta)} \cdot \frac{i_{1}(0)}{i_{2}(0)}
$$

and compare them with $\cos ^{2} \theta$ for examining the linear relationship. The results obtained are:

| $\theta$ | $0.0^{\circ}$ | $30.0^{\circ}$ | $45.0^{\circ}$ | $60.0^{\circ}$ | $90.0^{\circ}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\cos ^{2} \theta$ | 1.00 | 0.75 | 0.50 | 0.25 | 0.00 |
| $I(\theta) / I_{0}$ | 1.00 | 0.75 | 0.49 | 0.24 | 0.00 |



1. (c) Reflectivity measurement

The experimental arrangement shown below is used to determine the ratio of $I_{0}$ to $I_{1}$ which is proportional to the ratio of the reading $\left(i_{20}\right)$ of $D_{2}$ to the corresponding reading ( $i_{10}$ ) of $D_{1}$.


Then used the experimental arrangement shown below to measure the relativity $R_{p}$ of the sample at various incident angle $(\theta)$ while the incident light consists of p-component only. Let $i_{1}(\theta)$ and $i_{2}(\theta)$ be the readings of $D_{1}$ and $D_{2}$ respectively.


Then the reflectivity is

$$
R_{p}(\theta)=\frac{I(\theta)}{I_{0}}=\frac{i_{2}(\theta)}{i_{1}(\theta)} \cdot \frac{i_{10}}{i_{20}}
$$

Data and results:

$$
\begin{aligned}
& \psi_{1}=140.5^{\circ} \\
& i_{20}=19.8 \times 5 \mu \mathrm{~A} \\
& i_{10}=13.3 \mu \mathrm{~A}
\end{aligned}
$$

| $\theta\left({ }^{\circ}\right)$ | $i_{2}(\theta)$ | $i_{1}(\mu A)$ | $R_{p}(\theta)$ |
| :---: | :---: | :---: | :---: |
| 5 | $15.1 \times 0.2$ | 11.1 | 0.037 |
| 10 | $14.9 \times 0.2$ | 11.2 | 0.036 |
| 20 | $13.3 \times 0.2$ | 11.1 | 0.032 |
| 30 | $11.4 \times 0.2$ | 12.2 | 0.025 |
| 40 | $7.8 \times 0.2$ | 14.7 | 0.014 |
| 50 | $2.3 \times 0.2$ | 16.9 | 0.0037 |
| 53 | $0.7 \times 0.2$ | 11.3 | 0.0017 |
| 55 | $0.3 \times 0.2$ | 11.3 | 0.00059 |
| 56.3 (dark) | $\sim 0$ | 11.5 | $\sim 0$ |
| 58 | $0.3 \times 0.2$ | 11.5 | 0.0007 |
| 60 | $1.1 \times 0.2$ | 13.5 | 0.0024 |
| 64 | $6.5 \times 0.2$ | 16.7 | 0.011 |
| 66 | $7.8 \times 0.2$ | 11.8 | 0.018 |
| 68 | $16.3 \times 0.2$ | 15.0 | 0.029 |


| 72 | $5.3 \times 0.1$ | 11.7 | 0.061 |
| :---: | :---: | :---: | :---: |
| 76 | $13.1 \times 1$ | 14.0 | 0.13 |
| 80 | $4.4 \times 5$ | 11.7 | 0.25 |
| 84 | $9.1 \times 5$ | 14.5 | 0.42 |

The curve of reflectivity of p-component as a function of incident in plexiglass

2. The Brewster angle $\theta_{B}$ can be found from the above date as

$$
\theta_{B}=56.3^{\circ} \pm 0.2^{\circ}
$$

The index of refraction can be calculated as

$$
n=\tan \theta_{B}=1.50 \pm 0.01
$$

The sources of errors are:

1. Detector sensitivity is low.
2. The incident light does not consist of $p$-component only.
3. The degree scales are not uniform.

## EXPERIMENTAL PROBLEM 1: Grading Scheme(10 points)

Part 1. Reflectivity of the $\boldsymbol{p}$-component. 7 points, distributed as follows.
a. Determination of the transmission axis of the polarizer (A) in $p$-component measurement, 1 point.

| (Error less than $\pm 2^{\circ}$, | 1.0point; |
| :--- | :--- |
| error less than $\pm 3^{\circ}$, | 0.7 point; |
| error less than $\pm 4^{\circ}$, | 0.3 point; |
| error less than $\pm 5^{\circ}$, | 0.1 point.) |

b. Verification of the linearity of the light intensity detector(2 points). Draws the optical schematic diagram correctly, 1.0 point; (Without the correction of the fluctuation of the light intensity, 0.4 point only);

Uses $I / I_{0} \sim \cos ^{2} \theta$ figure to show the "linearity", 0.5 point;
Tabulate the measured data(with 5 points at least)correctly, 0.5 point.
c. Determination of the reflectivity of the p-component of the light as a function of incident angle, 4 points, distributed as follows.

Draws the optical schematic diagram correctly and tabulate the measured data perfectly, 2.0 points;
Plot the reflectivity as the function of incident angle with indication of errors, 2 points.

## Part 2. Determination of the refractive index of sample, 3 point.

Brewster angle of sample, 1 point;

| (Error less than $\pm 1^{\circ}$, | 1.0point; |
| :--- | :--- |
| error less than $\pm 2^{\circ}$, | 0.5 point; |
| error less than $\pm 3^{\circ}$, | 0.2 point; |
| error larger than $\pm 3^{\circ}$, | 0 point.) |
| The refractive index of sample, | 0.5 point. |
| Discussion and determination of errors, 1.5 points. |  |

## EXPERIMENTAL PROBLEM 2

## Black Box

Given a black box with two similar terminals. There are no more than three passive elements inside the black box. Find the values of elements in the equivalent circuit between the terminals. This box is not allowed to be opened.

## Experimental Apparatus

1. Double channel oscilloscope with a panel illustration, showing the name and function of each knob
2. Audio frequency signal generator with a panel illustration, showing the name and function of each knob
3. Resistance box with a fixed value of 100 ohm $(< \pm 0.5 \%)$
4. Several connecting wires
5. For the coaxial cables, the wire in black color at the terminal is grounded.
6. Log-log paper, semi-log paper, and millimeter paper are provided for use if necessary

Note: The knobs, which were not shown on the panel illustration of the "signal generator" and "oscilloscope", have been set to the correct positions. It should not be touched by the student.

## Experimental Requirements

1. Draw the circuit diagram in your experiment.
2. Show your measured data and the calculated results in the form of tables. Plot the experimental curves with the obtained results on the coordinate charts provided(indicate the title of the diagram and the titles and scale units of the coordinate axes)
3. Given the equivalent circuit of the black box and the names of the elements with their values in the equivalent circuit(write down the calculation formulas).

## Instructions

1. Do your experiment in the frequency range between 100 Hz and 50 kHz .
2. The output voltage of the signal generator should be less than 1.0 V (peak-to-peak). Set the "Out Attenuation" switch to " 20 " db position and it should not be changed.
3. On connecting the wires, be careful to manage the wiring so as to minimize the 50 Hz interference from the electric mains.

## Instruction for Using XD2 Type Frequency Generator

1. Set the "Out Attenuation" to " 20 " db position and it should not be changed.
2. Set the "Damping Switch" to "Fast" position.
3. The indication of the voltmeter of the signal generator is the relative value, but not the true value of the output.
4. Neglect the error of the frequency readings.

Note: For XD22 Type Audio Frequency generator, there is no "Damping Switch", and the "output" switch should be set to the sine " $\sim$ " position.

## Instruction for Using SS-5702 Type Oscilloscope

1. Keep the "V mode" switch in "Dual" position.
2. The "Volts/div" (black) and the "variable control" (red) vary the gain of the vertical amplifier, and when the "variable control" (red) is ill the fully clockwise position, the black setting are calibrated.
3. The "Times/div" (Black) varies the horizontal sweep rate from $0.5 \mu \mathrm{~s} / \mathrm{div}$ to $0.2 \mathrm{~s} / \mathrm{div}$, and they are calibrated when the "variable control" (red) is in the fully clockwise CAL position.
4. The "Trigging Source" (Trigging sweep signal) is used to select the trigging signal channel and the" level" control is used to adjust the amplitude of the trigging signal.
5. Measuring accuracy: $\pm 4 \%$.

## Instruction for Using "Resistance Box"

The resistance of the "Resistance Box" has been set to a value of 100ohm, and it should not be changed.

## Experimental problem 2...... <br> Solution

1. The circuit diagram is shown in Fig. 1


Fig. 1
We have the relation:

$$
\begin{gathered}
I=\frac{V_{R}}{R} ; \\
Z+R=\frac{V_{Z+R}}{I}=\frac{V_{Z+R}}{V_{R}} R
\end{gathered}
$$

2. Measure the values of $V_{Z+R}$ and $V_{R}$ at various frequencies ( $f$ ), the measured data and calculated value of $Z+R$ are shown in table l. "The $Z+R-f$ curve is plotted in Fig. 2


Table l. The magnitude of impedance verus frequency

| $f\left(\times 10^{3} \mathrm{~Hz}\right)$ | $U_{Z+R}\left(V_{p p}\right)$ | $U_{R} \mathrm{mV}_{p p}$ | $\mathrm{Z}+R\left(\times 10^{3} \Omega\right)$ |
| :---: | :---: | :---: | :---: |
| 0.100 | 0.600 | 22.0 | 2.73 |
| 0.200 | 0.600 | 45.0 | 1.33 |
| 0.400 | 0.600 | 94.0 | 0.638 |
| 0.700 | 0.300 | 92.0 | 0.326 |
| 0.900 | 0.300 | 121 | 0.248 |


| 1.00 | 0.300 | 136 | 0.220 |
| :--- | :--- | :--- | :--- |
| 1.10 | 0.300 | 140 | 0.214 |
| 1.16 | 0.300 | 141 | 0.213 |
| 1.25 | 0.300 | 140 | 0.214 |
| 1.50 | 0.300 | 120 | 0.250 |
| 2.00 | 0.300 | 88.0 | 0.341 |
| 4.00 | 0.300 | 78.0 | 0.769 |
| 8.00 | 0.600 | 38.0 | 1.58 |
| 15.0 | 0.600 | 20.0 | 3.00 |
| 30.0 | 0.600 | 10.0 | 6.00 |
| 50.0 | 0.600 | 6.0 | 10.0 |

From table 1 and Fig. 2, we got the conclusions:
(1) Current resonance (minimum of $Z$ ) occurs a $f_{0} \cong 1.16 \times 10^{3} \mathrm{~Hz}$.
(2) $f \ll f_{0}, Z \propto f, \Delta \varphi \approx-\pi / 2$. The impedance of the "black box" at low frequency is dominated by a inductance.
(3) $f\rangle>f_{0}, Z \propto f, \Delta \varphi \approx \pi / 2$. The impedance of the "black box" at high frequency is dominated by a inductance.
(4) Equivalent circuit of the "black box"; $r, L$ and $C$ connected in series shown in Fig. 3.


Fig. 3
3. Determination of the values of $r, L$ and $C$.
(a) $r$

At resonance frequency $f_{0}$

$$
V_{C}=-V_{L}
$$

Then

$$
Z+R=\frac{V_{Z+R}}{I}=\frac{V_{Z+R}}{V_{R}} R=r+R
$$

From table 1, $r+R=213 \Omega$, it is given $R=100 \Omega$, so the equivalent resistance $r$ in Fig. 3 is equal $113 \Omega$.
(b) $C$

At low frequency, $z_{L} \approx 0$ in Fig. 3. So the circuit could be considered as a series RC circuit.

From phasor diagram, Fig. 4,

$$
\frac{1}{\omega C}=Z_{C}=\frac{V_{C}}{I}=\frac{\sqrt{V_{Z+R}^{2}-V_{R+r}^{2}}}{I}
$$

Since $V_{R+r}^{2} / V_{Z+R}^{2} \approx 6 \times 10^{-3}$ at $f=100 \mathrm{~Hz}, V_{R+r}^{2}$ can beneglected with respect to $V_{Z+R}^{2}$, so

$$
\begin{aligned}
\frac{1}{\omega C} \approx \frac{V_{Z+R}}{I} & \approx Z+R=2.73 \times 10^{3} \Omega \\
C & \approx \frac{1}{\omega(Z+R)}=0.58 \mu f \\
C & \cong 0.58 \mu f
\end{aligned}
$$



Fig. 4
(c) $L$

At high frequency, $Z_{L} \approx 0$ in Fig. 3. So the circuit could be considered as a series RL circuit.

From phasor diagram, Fig. 5,

$$
\left|V_{L}\right|=\sqrt{V_{Z+R}^{2}-V_{r+R}^{2}},
$$

Since $V_{r+R}^{2} / V_{Z+R}^{2} \approx 4.5 \times 10^{-4}$ at $f=50 \mathrm{kHz}, V_{r+R}^{2}$ can be neglected with respect to $V_{Z+R}^{2}$, so


Fig. 5

$$
\begin{equation*}
\omega L=Z_{L}=\frac{V_{L}}{I}=\frac{\left|V_{Z+R}\right|}{I} \approx Z+R=10^{4} \Omega \tag{3}
\end{equation*}
$$

$$
L=\frac{Z+R}{\omega}=31.8 \mathrm{mH} .
$$

Error estimation:
It is given, precision of the resistance box reading $\Delta R / R \approx 0.5 \%$ precision of the voltmeter reading $\Delta V / V \approx 4 \%$
(1) Resistance $r$ : at resonance frequency $f_{0}$

$$
\begin{gathered}
r+R=\frac{V_{Z+R}}{V_{R}} R \\
\frac{\Delta(r+R)}{r+R}=\frac{\Delta V_{Z+R}}{V_{Z+R}}+\frac{\Delta V_{R}}{V_{R}}+\frac{\Delta R}{R} \approx 4 \%+4 \%+0.5 \%=8.5 \% \\
\Delta r=16 \Omega
\end{gathered}
$$

(2) Capacitance C: (Neglect the error of the frequency reading)

$$
\begin{gathered}
\frac{1}{\omega C} \cong Z_{C}=\frac{V_{Z+R}}{V_{R}} R \\
\frac{\Delta C}{C}=\frac{\Delta V_{Z+R}}{V_{Z+R}}+\frac{\Delta V_{R}}{V_{R}}+\frac{\Delta R}{R} \approx 8.8 \%
\end{gathered}
$$

The approximation $V_{C} \approx V_{Z+R}$ will introduce apercentageerror $0.3 \%$
(3) Inductance L: Similar to the results of capacitance C, but the percentage error introduced by the approximation $V_{L} \approx V_{Z+R}$ is much small ( $0.003 \%$ ) and thus negligible.

$$
\frac{\Delta L}{L} \approx 8.5 \% .
$$

## Experimental Problem 2: Grading Scheme (10 points maximum)

1. Measuring circuit is correct as shown in Fig.(a)


Fig. a
2. Correct data table and figure to show the characteristic of the black box $\cdots \cdot \cdot \cdot 2.0$ points
3. The equivalent circuit of the black box, and the names of the elements with their values in the equivalent circuit are correct
total 6.0 points
(a) R, L and C are connected in series
$\cdots \cdots 1.5$ point
(L and C are connected in series
$\cdots \cdots \cdot 1.0$ point)
(b) Correct value (error less than $15 \%$ ) for each element
$\cdots \cdots \cdot 0.5$ point ( $\times 3$ )
(error between $15 \%$ and $30 \% 0.3$ )
(error between $30 \%$ and $50 \%$ 0.1)
(c) Correct calculation formula for each element $\cdots \cdots \cdot 0.5$ point ( $\times 3$ )
(d) Error estimate is reasonable for each element $\cdots \cdots 0.5$ points $(\times 3)$

## Theoretical Question 1

This is essentially a question in special relativity. The core of the question is part (b) which involves a simulated experiment. It requires students to combine the concepts of gravitational red shifts, resonance absorption, Doppler shifts and the graphical interpretation of data.

Overall the question appears to have met its objective of allowing nearly all students to gain a few marks from part (a). A suprisingly large number of students were able to obtain essentially the correct solution to part (b) using the appropriate straight-line graph. Part (c) also produced many basically correct solutions with some of the best students simplifying their soloution to the logical limit. One student managed to obtain the correct answer making use of the 4 -momentum. The very best answers to this question were almost flawless and demonstrated a very high level of conceptual understanding and the ability to synthesise ideas from a number of different areas.

## Theoretical Question 2

This question is concerned with the propagation of waves in a medium with a varying refractive index and the different modes of propagation which occur. The responses to this question mirrored the marks distribution shown in Figure 1 for the overall theory results. A number of students gained near-perfect marks while an equivalent number gained very few. The most interesting part of the marking arose in connection with part (a), where the arc radius $R$ specified in the question needs to be established. The marking team encountered four distinguishable and valid approaches to establishing the result for $R$.

Part (c) proved to be a useful discriminator between those students who either did, or did not, realise that a seris of paths, or modes, exists from the source to the receiver. The numerical estimates in part (d), and intended to assist the markers, required some care in marking according to the way in which students treated the issue of significant figures during the calculation. Part (e), which led to the conclusion that the ray with the smallest calue of initial angle will arrive first, was a useful discriminator.

## Theoretical Question 3

This question is essentially a problem in mechanics with elements of hydrostatics. It involves the concepts of Archimedes' Principle, small oscillations and rotational dynamics applied to an interesting geometry.

One common mistake of interpretation noted by the examiners was to set the length of the rod equal to the radius rather than to the diameter of the cylinder. In line with the policy on marking, students were only penalised once for this mistake provided that the rest of their analysis was consistent with this assumption. The clever aspect of the problem was in part (d) where some students attempted to estimate the solution to the transcendental equation $\alpha-\sin \alpha \cos \alpha=1.61 \sin \alpha$, rather than simply checking that $\alpha \simeq 1.57(\pi / 2)$ gave a reasonable result. Students from two teams used numerical methods to obtain a more precise value for $\alpha$. One student who correctly applied Newton's method to solve the equation for $\alpha$ received the special prize for mathematics.

## Experimental Question 1

This question was concerned with the motion of small objects (cylinders) in a viscous medium, and was designed to test as wide a range of experimental skills as possible. In particular the question aimed to test:

- understanding of the concept of terminal velocity.
- experimental technique; the experiment required careful hand-eye coordination to reduce systematic effects (for example by dropping the cylinders each time with the same orientation and using multiple timings to reduce the scatter in the results).
- the ability to graph and interpret data including the use of logarithmic and linear plots and the interpretation of slopes and intercepts.
- estimation of uncertainties in the results.

The experiment generally worked as expected. Experimental techniques were uniformly good, and the students demonstrated excellent manipulative skills. Their main weakness was in the handling of the determination of the density of the glycerine from the graph of fall time as a function of the density of the cylinders. Students in general did not measure the intercept on the density axis but calculated the density from the intercept on the fall time axis and the slope of the graph.

## Experimental Question 2

This question made use of a laser pointer to carry out several experiments in optics. The first task concerned the use of a metal ruler as a diffraction grating. In this experiment the diffraction pattern was formed by reflection with the incident laser beam at nearly normal incidence to the ruler. (This geometry is rather different from the more common demonstration where the incident beam is at close to grazing incidence.) A number of students had difficulty with this geometry and failed to obtain a convincing pattern.

The second experiment investigated the reflection and transmission of light through transparent media. The main difficulty with the measurements was that changes in intensity had to be estimated by eye using a set of calibrated transmission discs. This was much more demanding than using, for example, a photodiode and multimeter as it required the exercise of considerable experimental judgement. It therefore provided an excellent test of a student's experimental technique.

The final experiment was concerned with the scattering of light from a translucent material formed by adding a few drops of milk to water. The amount of scattering and the reduction in the transmitted intensity were measured as a function of the concentration of milk. Students had considerable difficulty with this experiment with some not recognising the phenomena they were meant to be observing. However the best students were still able to obtain convincing results. The exercise therefore provided good discrimination between the most able students.

## Theoretical Question 1

## Gravitational Red Shift and the Measurement of Stellar Mass

(a) (3 marks)

A photon of frequency $f$ possesses an effective inertial mass $m$ determined by its energy. Assume that it has a gravitational mass equal to this inertial mass. Accordingly, a photon emitted at the surface of a star will lose energy when it escapes from the star's gravitational field. Show that the frequency shift $\Delta f$ of the photon when it escapes from the surface of the star to infinity is given by

$$
\frac{\Delta f}{f} \simeq-\frac{G M}{R c^{2}}
$$

for $\Delta f \ll f$ where:

- $G=$ gravitational constant
- $R=$ radius of the star
- $c=$ velocity of light
- $M=$ mass of the star.

Thus, the red-shift of a known spectral line measured a long way from the star can be used to measure the ratio $M / R$. Knowledge of $R$ will allow the mass of the star to be determined.
(b) (12 marks)

An unmanned spacecraft is launched in an experiment to measure both the mass $M$ and radius $R$ of a star in our galaxy. Photons are emitted from $\mathrm{He}^{+}$ions on the surface of the star. These photons can be monitored through resonant absorption by $\mathrm{He}^{+}$ions contained in a test chamber in the spacecraft. Resonant absorption accors only if the $\mathrm{He}^{+}$ions are given a velocity towards the star to allow exactly for the red shifts.
As the spacecraft approaches the star radially, the velocity relative to the star $(v=\beta c)$ of the $\mathrm{He}^{+}$ ions in the test chamber at absorption resonance is measured as a function of the distance $d$ from the (nearest) surface of the star. The experimental data are displayed in the accompanying table.
Fully utilize the data to determine graphically the mass $M$ and radius $R$ of the star. There is no need to estimate the uncertainties in your answer.

## Data for Resonance Condition

| Velocity parameter | $\beta=v / c\left(\times 10^{-5}\right)$ | 3.352 | 3.279 | 3.195 | 3.077 | 2.955 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Distance from surface of star | $d\left(\times 10^{8} \mathrm{~m}\right)$ | 38.90 | 19.98 | 13.32 | 8.99 | 6.67 |

(c) (5 marks)

In order to determine $R$ and $M$ in such an experiment, it is usual to consider the frequency correction due to the recoil of the emitting atom. [Thermal motion causes emission lines to be broadened without displacing emission maxima, and we may therefore assume that all thermal effects have been taken into account.]
(i) (4 marks)

Assume that the atom decays at rest, producing a photon and a recoiling atom. Obtain the relativistic expression for the energy $h f$ of a photon emitted in terms of $\Delta E$ (the difference in rest energy between the two atomic levels) and the initial rest mass $m_{0}$ of the atom.
(ii) (1 mark)

Hence make a numerical estimate of the relativistic frequency shift $\left(\frac{\Delta f}{f}\right)_{\text {recoil }}$ for the case of $\mathrm{He}^{+}$ions.
Your answer should turn out to be much smaller than the gravitational red shift obtained in part (b).

Data:

| Velocity of light | $c$ | $=3.0 \times 10^{8} \mathrm{~ms}^{-1}$ |
| :--- | :--- | :--- |
| Rest energy of He | $m_{0} c^{2}=4 \times 938(\mathrm{MeV})$ |  |
| Bohr energy | $E_{n}=-\frac{13.6 Z^{2}}{n^{2}}(\mathrm{eV})$ |  |
| Gravitational constant | $G$ | $=6.7 \times 10^{-11} \mathrm{Nm}^{2} \mathrm{~kg}^{-2}$ |

## Theoretical Question 2

## Sound Propagation

## Introduction

The speed of propagation of sound in the ocean varies with depth, temperature and salinity. Figure 1 (a) below shows the variation of sound speed $c$ with depth $z$ for a case where a minimum speed value $c_{0}$ occurs midway betweeen the ocean surface and the sea bed. Note that for convenience $z=0$ at the depth of this sound speed minimum, $z=z_{S}$ at the surface and $z=-z_{b}$ at the sea bed. Above $z=0, c$ is given by

$$
c=c_{0}+b z
$$

Below $z=0, c$ is given by

$$
c=c_{0}-b z
$$

In each case $b=\left|\frac{d c}{d z}\right|$, that is, $b$ is the magnitude of the sound speed gradient with depth; $b$ is assumed constant.


Figure 1 (a)


Figure 1 (b)

Figure 1(b) shows a section of the $z-x$ plane through the ocean, where $x$ is a horizontal direction. The variation of $c$ with respect to $z$ is shown in figure 1(a). At the position $z=0, x=0$, a sound source $S$ is located. A 'sound ray' is emitted from $S$ at an angle $\theta_{0}$ as shown. Because of the variation of $c$ with $z$, the ray will be refracted.
(a) (6 marks)

Show that the trajectory of the ray, leaving the source $S$ and constrained to the $z-x$ plane forms an arc of a circle with radius $R$ where

$$
R=\frac{c_{0}}{b \sin \theta_{0}} \text { for } 0 \leq \theta_{0}<\frac{\pi}{2}
$$

(b) (3 marks)

Derive an expression involving $z_{S}, c_{0}$ and $b$ to give the smallest value of the angle $\theta_{0}$ for upwardly directed rays which can be transmitted without the sound wave reflecting from the sea surface.
(c) (4 marks)

Figure 1(b) shows the position of a sound receiver $H$ which is located at the position $z=0, x=X$. Derive an expression involving $b, X$ and $c_{0}$ to give the series of angles $\theta_{0}$ required for the sound ray emerging from $S$ to reach the receiver $H$. Assume that $z_{S}$ and $z_{b}$ are sufficiently large to remove the possibility of reflection from sea surface or sea bed.
(d) (2 marks)

Calculate the smallest four values of $\theta_{0}$ for refracted rays from $S$ to reach $H$ when

- $X=10000 \mathrm{~m}$
- $c_{0}=1500 \mathrm{~ms}^{-1}$
- $b=0.02000 \mathrm{~s}^{-1}$
(e) (5 marks)

Derive an expression to give the time taken for sound to travel from $S$ to $H$ following the ray path associated with the smallest value of angle $\theta_{0}$, as determined in part (c). Calculate the value of this transit time for the conditions given in part (d). The following result may be of assistance:

$$
\int \frac{d x}{\sin x}=\ln \tan \left(\frac{x}{2}\right)
$$

Calculate the time taken for the direct ray to travel from $S$ to $H$ along $z=0$. Which of the two rys will arrive first, the ray for which $\theta_{0}=\pi / 2$, or the ray with the smallest value of $\theta_{0}$ as calculated for part (d)?

## Theoretical Question 3

## Cylindrical Buoy

(a) (3 marks)

A buoy consists of a solid cylinder, radius $a$, length $l$, made of lightweight material of uniform density $d$ with a uniform rigid rod protruding directly outwards from the bottom halfway along the length. The mass of the rod is equal to that of the cylinder, its length is the same as the diameter of the cylinder and the density of the rod is greater than that of seawater. This buoy is floating in sea-water of density $\rho$.
In equilibrium derive an expression relating the floating angle $\alpha$, as drawn, to $d / \rho$. Neglect the volume of the rod.

(b) (4 marks)

If the buoy, due to some perturbation, is depressed vertically by a small amount $z$, it will experience a nett force, which will cause it to begin oscillating vertically about the equilibtium floating position. Determine the frequencty of this vertical mode of vibration in terms of $\alpha, g$ and $a$, where $g$ is the acceleration due to gravity. Assume the influence of water motion on the dynamics of the buoy is such as to increase the effective mass of the buoy by a factor of one third. You may assume that $\alpha$ is not small.

(c) (8 marks)

In the approximation that the cylinder swings about its horizontal central axis, determine the frequency of swing again in terms of $g$ and $a$. Neglect the dynamics and viscosity of the water in this case. The angle of swing is assumed to be small.

(d) (5 marks)

The buoy contains sensitive acelerometers which can measure the vertical and swinging motions and can relay this information by radio to shore. In relatively calm waters it is recorded that the vertical oscillation period is about 1 second and the swinging oscillation period is about 1.5 seconds. From this information, show that the floating angle $\alpha$ is about $90^{\circ}$ and thereby estimate the radius of the buoy and its total mass, given that the cylinder length $l$ equals $a$.
[You may take it that $\rho \simeq 1000 \mathrm{kgm}^{-3}$ and $g \simeq 9.8 \mathrm{~ms}^{-2}$.]

## Original Theoretical Question 3

The following question was not used in the XXVI IPhO examination.

## Laser and Mirror

(a)

Light of frequency $f_{i}$ and speed $c$ is directed at an angle of incidence $\theta_{i}$ to the normal of a mirror, which is receding at speed $u$ in the direction of the normal. Assuming the photons in the light beam undergo an elastic collision in the rest frame of the mirror, determine in terms of $\theta_{i}$ and $u / c$ the angle of reflection $\theta_{r}$ of the light and the reflected frequency $f_{r}$, with respect to the original frame.

[You may assume the following Lorentz transformation rules apply to a particle with energy $E$ and momentum $\mathbf{p}$ :

$$
p_{\perp}=p_{\perp}, \quad p_{\|}=\frac{p_{\|}-v E / c^{2}}{\sqrt{1-v^{2} / c^{2}}}, \quad E=\frac{E-v p_{\|}}{\sqrt{1-v^{2} / c^{2}}}
$$

where $\mathbf{v}$ is the relative velocity between the two inertial frames; $p$ stands for the component of momentum perpendicular to $\mathbf{v}$ and $p$ represents the component of momentum parallel to $\mathbf{v}$.]
(b)

A thin rectangular light mirror, perfectly reflecting on each side, of width $2 a$ and mass $m$, is mounted in a vacuum (to eliminate air resistance), on essentially frictionless needle bearings, so that it can rotate about a vertical axis. A narrow laser beam operating continuously with power $P$ is incident on the mirror, at a distance $b$ from the axis, as drawn.

elevation

plan

Suppose the mirror is originally at rest. The impact of the light causes the mirror to acquire a very small but not constant angular acceleration. To analyse the siuation approximately, assume that at any given stage in the acceleration process the angular velocity $\omega$ of the mirror is constant throughout any one complete revolution, but takes on a slightly larger value in the next revolution due to the angular momentum imparted to the mirror by the light during the preceding revolution. Ignoring second order terms in the ratio (mirror velocity $/ c$ ), calculate this increment of angular momentum per revolution at any given value of $\omega$. [HINT: You may find it useful to know that $\int \sec ^{2} \theta d \theta=\tan \theta$.]
(c)

Using the fact that the velocity of recoil of the mirror remains small compared with $c$, derive an approximate expression for $\omega$ as a function of time.
(d)

As the mirror rotates, there will be instants when the light is reflected from its edge, giving the reflected ray an angle of somewhat more than $90^{\circ}$ with respect to the incident beam.. A screen 10 km away, with its normal perpendicular to the incident beam, intercepts the beam reflected from near the mirror's edge. Find the deviation $\xi$ of that extreme spot from its initial position (as shown by the dashed line, when the mirror was almost at rest), after the laser has operated for 24 hours. You may suppose the laser power is $P=100 \mathrm{~W}$, that the mirror has mass $m=1$ gram and that the geometry of the apparatus corresponds to $a=b \sqrt{2}$. Neglect diffraction effects at the edge.


## Experimental Question 1

## Terminal velocity in a viscous liquid

An object falling in a liquid will eventually reach a constant velocity, called the terminal velocity. The aim of this experiment is to measure the terminal velocities of objects falling through glycerine.

For a sphere of radius $r$ falling at velocity $v$ through a viscous liquid, the viscous force $F$ is given by $F=6 \pi \eta r v$. Here $\eta$ is a property of the liquid called the viscosity. In this experiment you will measure the terminal velocity of metal cylinders (because cylinders are easier to make than spheres). The diameter of each cylinder is equal to its length, and we will assume the viscous force on such a cylinder is similar to the viscous force on a sphere of the same diameter, $2 r$ :

$$
\begin{equation*}
F_{c y l}=6 \pi \kappa \eta r^{m} v \tag{1}
\end{equation*}
$$

where $\kappa=1, m=1$ for a sphere.

## Preliminary

Calculation of terminal velocity (2 marks)
If $\rho$ is the density of the culinder and $\rho^{\prime}$ is the density of the liquid, show that the terminal velocity $v_{T}$ of the cylinder is given by

$$
\begin{equation*}
v_{T}=C r^{3-m}\left(\rho-\rho^{\prime}\right) \tag{2}
\end{equation*}
$$

where $C$ is a constant and derive a expression for $C$.

## Experiment

Use the equipment available to determine the numerical value of the exponent $m$ ( 10 marks) and the density of glycerine (8 marks).

## Notes

- For consistency, try to ensure that the cylinders fall in the same orientation, with the axis of the cylinder horizontal.
- The tolerances on the diameter and the length of the cylinders are 0.05 mm (you need not measure them yourself).
- There is a brass sieve inside the container that you should use to retrieve the metal cylinders. Important: make sure the sieve is in place before dropping objects into the glycerine, otherwise you will not be able to retrieve them for repeat measurements.
- When glycerine absorbs water from the atmosphere, it becomes less viscous. Ensure that the cylinder of glycerine is covered with the plastic film provided when not in use.
- Do not mix cylinders of different size and different material after the experiment.
Material
Aluminium
Titanium
Stainless steel
Copper

Density ( $\mathrm{kgm}^{-3}$ )
$2.70 \times 10^{3}$
$4.54 \times 10^{3}$
$7.87 \times 10^{3}$
$8.96 \times 10^{3}$

## Experimental Question 2

## Diffraction and Scattering of Laser Light

The aim of this experiment is to demonstrate and quantify to some extent the reflection, diffraction, and scattering of light, using visible radiation from a Laser Diode source. A metal ruler is employed as a diffraction grating, and a perspex tank, containing water and diluted milk, is used to determine reflection and scattering phenomena.

## Section 1 (6 marks)

Place the 150 mm length metal ruler provided so that it is nearly normal to the incident laser beam, and so that the laserr beam illuminates several rulings on it. Observe a number of "spots" of light on the white paper screen provided, caused by the phenomenon of diffraction.

Draw the overall geometry you have employed and measure the position and separation of these spots with the screen at a distance of approximately 1.5 metres from the ruler.

Using the relation

$$
N \lambda=h \sin \beta
$$

where $\quad N$ is the order of diffraction
$\lambda$ is the radiation wavelength
$h$ is the grating period
$\beta$ is the angle of diffraction
and the information obtained from your measurements, determine the wavelength of the laser radiation.

## Section $2(4$ marks)

Now insert the empty perspex tank provided into the space between the laser and the white paper screen. Set the tank at approximately normal incidence to the laser beam.
(i) Observe a reduction in the emergent beam intensity, and estimate the percentage value of this reduction. Some calibrated transmission discs are provided to assist with this estimation. Remember that the human eye has a logarithmic response.

This intensity reduction is caused primarily by reflection losses at the aid/perspex boundaries, of which there are four in this case. THe reflection coefficient for normal incidence at each boundary, $R$, which is the ratio of the reflectied to incident intensities, is given by

$$
R=\left\{\left(n_{1}-n_{2}\right) /\left(n_{1}+n_{2}\right)\right\}^{2}
$$

where $n_{1}$ and $n_{2}$ are the refractive indices before and after the boundary. The corresponding transmission coefficient, assuming zero absorption in the perspex, is fiven by

$$
T=1-R
$$

(ii) Assuming a refractive index of 1.59 for the perspex and neglecting the effect of multiple reflections and cogerence, calculate the intensity transmission coefficient of the empty perspex tank. Compare this result with the estimate you made in Part (i) of this Section.

## Section 3 (1 mark)

Without moving the perspex tank, repeat the observations and calculations in Section 2 with the 50 mL of water provided in a beaker now added to the tank. Assume the refractive index of water to be 1.33.

Section 4 (10 marks)
(i) Add 0.5 mL ( 12 drops) of milk (the scattering material) to the 50 mL of water in the perspex tank, and stir well. Measure as accurately as possible the total angle through which the laser light is scattered, and the diameter of the emerging light patch at the exit face of the tank, noting that these quantities are related. Also estimate the reduction in transmitted intensity, as in earlier sections.
(ii) Add a further 0.5 mL of milk to the tank, and repeat the measurements requested in part (i).
(iii) Repeat the process in part (ii) until very little or no transmitted laser light can be observed.
(iv) Determine the relationship between scattering angle and milk concentration in the tank.
(v) Use your results, and the relationship

$$
I=I_{0} e^{-\mu z}=T_{m i l k} \times I_{0}
$$

where

| $I_{0}$ | is the input intensity |
| :---: | :--- |
| $I$ | is the emerging intensity |
| $z$ | is the distance in the tank |
| $\mu$ | is the attenuation coefficient and equals a constant times the concentration of the scatterer |
| $T_{m i l k}$ | is the transmission coefficient for the milk |

to obtain an estimate for the value of $\mu$ for a scatterer concentration of $10 \%$.

## Solutions to Theoretical Question 1

## Gravitational Red Shift and the Measurement of Stellar Mass

(a)

If a photon has an effective inertial mass $m$ determined by its energy then $m c^{2}=h f$ or $m=\frac{h f}{c^{2}}$. Now, assume that gravitational mass $=$ inertial mass, and consider a photon of energy $h f$ (mass $m=\frac{h f}{c^{2}}$ ) emitted upwards at a distance $r$ from the centre of the star. It will lose energy on escape from the gravitational field of the star.
Apply the principle of conservation of energy:
Change in photon energy $\left(h f_{i}-h f_{f}\right)=$ change in gravitational energy, where subscript $i \rightarrow$ initial state and subscript $f \rightarrow$ final state.

$$
\begin{aligned}
h f_{i}-h f_{f} & =-\frac{G M m_{f}}{\infty}-\left[-\frac{G M m_{i}}{r}\right] \\
h f_{f} & =h f_{i}-\frac{G M m_{i}}{r} \\
h f_{f} & =h f_{i}-\frac{G M \frac{h f_{i}}{c^{2}}}{r} \\
h f_{f} & =h f_{i}\left[1-\frac{G M}{r c^{2}}\right] \\
\frac{f_{f}}{f_{i}} & =\left[1-\frac{G M}{r c^{2}}\right] \\
\frac{\Delta f}{f} & =\frac{f_{f}-f_{i}}{f_{i}}=-\frac{G M}{r c^{2}}
\end{aligned}
$$

The negative sign shows red-shift, i.e. a decrease in $f$, and an increase in wavelength.
Thus, for a photon emitted from the surface of a star of radius $R$, we have

$$
\frac{\Delta f}{f}=\frac{G M}{R c^{2}}
$$

Since the change in photon energy is small, $(\delta f \ll f)$,

$$
m_{f} \simeq m_{i}=\frac{h f_{i}}{c^{2}}
$$

(b)

The change in photon energy in ascending from $r_{i}$ to $r_{f}$ is given by

$$
\begin{aligned}
h f_{i}-h f_{f} & =-\frac{G M m_{f}}{r_{f}}+\frac{G M m_{i}}{r_{i}} \\
& \simeq \frac{G M h f_{i}}{c^{2}}\left[\frac{1}{r_{i}}-\frac{1}{r_{f}}\right] \\
\therefore \frac{f_{f}}{f_{i}} & =1-\frac{G M}{c^{2}}\left[\frac{1}{r_{i}}-\frac{1}{r_{f}}\right]
\end{aligned}
$$

In the experiment, $R$ is the radius of the star, $d$ is the distance from the surface of the star to the spacecraft and the above equation becomes:

$$
\begin{equation*}
\frac{f_{f}}{f_{i}}=1-\frac{G M}{c^{2}}\left[\frac{1}{R}-\frac{1}{R+d}\right] \tag{1}
\end{equation*}
$$

The frequency of the photon must be doppler shifted back from $f_{f}$ to $f_{i}$ in order to cause resonance excitation of the $\mathrm{He}^{+}$ions in the spacecraft.
Thus apply the relativistic Doppler principle to obtain:

$$
\frac{f^{\prime}}{f_{f}}=\sqrt{\frac{1+\beta}{1-\beta}}
$$

where $f^{\prime}$ is the frequency as received by $\mathrm{He}^{+}$ions in the spacecraft, and $\beta=v / c$. That is, the gravitationally reduced frequency $f_{f}$ has been increased to $f^{\prime}$ because of the velocity of the ions on the spacecraft towards the star. Since $\beta \ll 1$,

$$
\frac{f_{f}}{f^{\prime}}=(1-\beta)^{\frac{1}{2}}(1+\beta)^{-\frac{1}{2}} \simeq 1-\beta
$$

Alternatively, since $\beta \ll 1$, use the classical Doppler effect directly.
Thus

$$
f^{\prime}=\frac{f_{f}}{1-\beta}
$$

or

$$
\frac{f_{f}}{f^{\prime}}=1-\beta
$$

Since $f^{\prime}$ must be equal to $f_{i}$ for resonance absorption, we have

$$
\begin{equation*}
\frac{f_{f}}{f_{i}}=1-\beta \tag{2}
\end{equation*}
$$

Substitution of 2 into 1 gives

$$
\begin{equation*}
\beta=\frac{G M}{c^{2}}\left(\frac{1}{R}-\frac{1}{R+d}\right) \tag{3}
\end{equation*}
$$

Given the experimental data, we look for an effective graphical solution. That is, we require a linear equation linking the experimental data in $\beta$ and $d$.
Rewrite equation 3:

$$
\beta=\frac{G M}{c^{2}}\left[\frac{R+d-\boldsymbol{R}}{(R+d) R}\right]
$$

Inverting the equation gives:

$$
\frac{1}{\beta}=\left(\frac{R c^{2}}{G M}\right)\left[\frac{R}{d}+1\right]
$$

or

$$
\frac{1}{\beta}=\left(\frac{R^{2} c^{2}}{G M}\right) \frac{1}{d}+\frac{R c^{2}}{G M}
$$

Graph of $\frac{1}{\beta}$ vs. $\frac{1}{d}$


The slope is $\left(\frac{R c^{2}}{G M}\right) R=\alpha R$
The $\frac{1}{\beta}$-intercept is $\left(\frac{R c^{2}}{G M}\right)=\alpha$

$$
\begin{equation*}
\text { and the } \frac{1}{d} \text {-intercept is }-\frac{1}{R} \tag{B}
\end{equation*}
$$

$R$ and $M$ can be conveniently determined from (A) and (B). Equation (C) is redundant. However, it may be used as an (inaccurate) check if needed.
From the given data:

$$
\begin{array}{r}
R=1.11 \times 10^{8} \mathrm{~m} \\
M=5.2 \times 10^{3} 0 \mathrm{~kg}
\end{array}
$$

$$
\begin{array}{r}
\text { From the graph, the slope } \quad \alpha R=3.2 \times 10^{12} \mathrm{~m} \\
\text { The } \frac{1}{\beta} \text {-intercept } \quad \alpha=\frac{R c^{2}}{G M}=0.29 \times 10^{5} \tag{B}
\end{array}
$$

Dividing (A) by (B)

$$
R=\frac{3.2 \times 10^{12} \mathrm{~m}}{0.29 \times 10^{5}} \simeq 1.104 \times 10^{8} \mathrm{~m}
$$

Substituting this value of $R$ back into (B) gives:

$$
M=\frac{R c^{2}}{g \alpha}=\frac{\left(1.104 \times 10^{8}\right) \times\left(3.0 \times 10^{8}\right)^{2}}{\left(6.7 \times 10^{-11}\right) \times\left(0.29 \times^{1} 0^{5}\right)}
$$

or $M=5.11 \times 10^{30} \mathrm{~kg}$
(c)
(i)

## Atom before the decay Atom and photon after the decay



For the photon, photon momentum is $p=\frac{h f}{c}$ and photon energy is $E=h f$.
Use the mass-energy equivalence, $E=m c^{2}$, to relate the internal energy change of the atom to the rest-mass change. Thus:

$$
\begin{equation*}
\Delta E=\left(m_{0}=m_{0}^{\prime}\right) c^{2} \tag{1}
\end{equation*}
$$

In the laboratory frame of reference the energy before emission is

$$
\begin{equation*}
E=m_{0} c^{2} \tag{2}
\end{equation*}
$$

Recalling the relativistic relation

$$
E^{2}=p^{2} c^{2}+m_{0}^{2} c^{4}
$$

The energy after emission of a photon is

$$
\begin{equation*}
E=\sqrt{p^{2} c^{2}+m_{0}^{\prime 2} c^{4}}+h f \tag{3}
\end{equation*}
$$

where also $p=h f / c$ by conservation of momentum.
Conservation of energy requires that $(2)=(3)$, so that:

$$
\begin{array}{r}
\left(m_{0} c^{2}-h f\right)^{2}=(h f)^{2}+m_{0}^{2} c^{4} \\
\left(m_{0} c^{2}\right)^{2}-2 h f m_{0} c^{2}=m_{0}^{2} c^{4}
\end{array}
$$

Carrying out the algebra and using equation (1):

$$
\begin{aligned}
h f\left(2 m_{0} c^{2}\right) & =\left(m_{0}^{2}-m_{0}^{\prime 2}\right) c^{4} \\
& =\left(m_{0}-m_{0}^{\prime}\right) c^{2}\left(m_{0}+m_{0}^{\prime}\right) c^{2} \\
& =\Delta E\left[2 m_{0}-\left(m_{0}-m_{0}^{\prime}\right)\right] c^{2} \\
& =\Delta E\left[2 m_{0} c^{2}-\Delta E\right]
\end{aligned}
$$

$$
h f=\Delta E\left[1-\frac{\Delta E}{2 m_{0} c^{2}}\right]
$$

(ii)

For the emitted photon,

$$
h f=\Delta E\left[1-\frac{\Delta E}{2 m_{0} c^{2}}\right] .
$$

If relativistic effects are ignored, then

$$
h f_{0}=\Delta E
$$

Hence the relativistic frequency shift $\frac{\Delta f}{f_{0}}$ is given by

$$
\frac{\Delta f}{f_{0}}=\frac{\Delta E}{2 m_{0} c^{2}}
$$

For $\mathrm{He}^{+}$transition $(n=2 \rightarrow 1)$, applying Bohr theory to the hydrogen-like helium ion gives:

$$
\Delta E=13.6 \times 2^{2} \times\left[\frac{1}{1^{2}}-\frac{1}{2^{2}}\right]=40.8 \mathrm{ev}
$$

Also, $m_{0} c^{2}=3.752 \times 10^{6} \mathrm{eV}$. Therefore the frequency shift due to the recoil gives

$$
\frac{\Delta f}{f_{0}} \simeq 5.44 \times 10^{-12}
$$

This is very small compared to the gravitational red-shift of $\frac{\Delta f}{f} \sim 10^{-5}$, and may be ignored in the gravitational red-shift experiment.

## Solutions to Theoretical Question 2

(a)

Snell's Law may be expressed as

$$
\begin{equation*}
\frac{\sin \theta}{\sin \theta_{0}}=\frac{c}{c_{0}} \tag{1}
\end{equation*}
$$

where $c$ is the speed of sound.
Consider some element of ray path $d s$ and treat this as, locally, an arc of a circle of radius $R$. Note that $R$ may take up any value between 0 and $\infty$. Consider a ray component which is initially directed upward from $S$.


In the diagram, $d s=R d \theta$, or $\frac{d s}{d \theta}=R$.
From equation (1), for a small change in speed $d c$,

$$
\cos \theta d \theta=\frac{\sin \theta_{0}}{c_{0}} d c
$$

For the upwardly directed ray $c=c_{0}+b z$ so $d c=b d z$ and

$$
\frac{\sin \theta_{0}}{c_{0}} b d z=\cos \theta d \theta, \text { hence } d z=\frac{c_{0}}{\sin \theta_{0}} \frac{1}{b} \cos \theta d \theta .
$$

We may also write (here treating $d s$ as straight) $d z=d s \cos \theta$. So

$$
d s=\frac{c_{0}}{\sin \theta_{0}} \frac{1}{b} d \theta
$$

Hence

$$
\frac{d s}{d \theta}=R=\frac{c_{0}}{\sin \theta_{0}} \frac{1}{b} .
$$

This result strictly applies to the small arc segments $d s$. Note that from equation (1), however, it also applies for all $\theta$, i.e. for all points along the trajectory, which therefore forms an arc of a circle with radius $R$ until the ray enters the region $z<0$.
(b)


Here

$$
\begin{aligned}
z_{s} & =R-R \sin \theta_{0} \\
& =R\left(1-\sin \theta_{0}\right) \\
& =\frac{c_{0}}{b \sin \theta_{0}}\left(1-\sin \theta_{0}\right),
\end{aligned}
$$

from which

$$
\theta_{0}=\sin ^{-1}\left[\frac{c_{0}}{b z_{s}+c_{0}}\right] .
$$

(c)


The simplest pathway between $S$ and $H$ is a single arc of a circle passing through $S$ and $H$. For this pathway:

$$
X=2 R \cos \theta_{0}=\frac{2 c_{0} \cos \theta_{0}}{b \sin \theta_{0}}=\frac{2 c_{0}}{b} \cot \theta_{0}
$$

Hence

$$
\cot \theta_{0}=\frac{b X}{2 c_{0}} .
$$

The next possibility consists of two circular arcs linked as shown.


For this pathway:

$$
\frac{X}{2}=2 R \cos \theta_{0}=\frac{2 c_{0}}{b} \cot \theta_{0}
$$

i.e.

$$
\cot \theta_{0}=\frac{b X}{4 c_{0}}
$$

In general, for values of $\theta_{0}<\frac{\pi}{2}$, rays emerging from $S$ will reach $H$ in $n$ arcs for launch angles given by

$$
\theta_{0}=\cot ^{-1}\left[\frac{b X}{2 n c_{0}}\right]=\tan ^{-1}\left[\frac{2 n c_{0}}{b X}\right]
$$

where $n=1,2,3,4, \ldots$
Note that when $n=\infty, \theta_{0}=\frac{\pi}{2}$ as expected for the axial ray.
(d)

With the values cited, the four smallest values of launch angle are

| $n$ | $\theta_{0}$ (degrees) |
| :---: | :---: |
| 1 | 86.19 |
| 2 | 88.09 |
| 3 | 88.73 |
| 4 | 89.04 |

(e)

The ray path associated with the smallest launch angle consists of a single arc as shown:
(2)


We seek

$$
\int_{1}^{3} d t=\int_{1}^{3} \frac{d s}{c}
$$

Try first:

$$
t_{12}=\int_{1}^{2} \frac{d s}{c}=\int_{\theta_{0}}^{\pi / 2} \frac{R d \theta}{c}
$$

Using

$$
R=\frac{c}{b \sin \theta}
$$

gives

$$
t_{12}=\frac{1}{b} \int_{\theta_{0}}^{\pi / 2} \frac{d \theta}{\sin \theta}
$$

so that

$$
t_{12}=\frac{1}{b}\left[\ln \tan \frac{\theta}{2}\right]_{\theta_{0}}^{\pi / 2}=-\frac{1}{b} \ln \tan \frac{\theta_{0}}{2}
$$

Noting that $t_{13}=2 t_{12}$ gives

$$
t_{13}=-\frac{2}{b} \ln \tan \frac{\theta_{0}}{2} .
$$

For the specified $b$, this gives a transit time for the smallest value of launch angle cited in the answer to part (d), of

$$
t_{13}=6.6546 \mathrm{~s}
$$

The axial ray will have travel time given by

$$
t=\frac{X}{c_{0}}
$$

For the conditions given,

$$
t_{13}=6.6666 \mathrm{~s}
$$

thus this axial ray travels slower than the example cited for $n=1$, thus the $n=1$ ray will arrive first.

## Solutions to Theoretical Question 3

(a)

The mass of the rod is given equal to the mass of the cylinder $M$ which itself is $\pi a^{2} l d$. Thus the total mass equals $2 M=2 \pi a^{2} l d$. The mass of the displaced water is surely less than $\pi a^{2} l \rho$ (when the buoy is on the verge of sinking). Using Archimedes' principle, we may at the very least expect that

$$
2 \pi a^{2} l d<\pi a^{2} l \rho \text { or } d<\rho / 2
$$

In fact, with the floating angle $\alpha(<\pi)$ as drawn, the volume of displaced water is obtained by geometry:


$$
V=l a^{2} \alpha-l a^{2} \sin \alpha \cos \alpha .
$$

By Archimedes' principle, the mass of the buoy equals the mass of displaced water. Therefore, $2 \pi a^{2} l d=l a^{2} \rho(\alpha-\sin \alpha \cos \alpha)$, i.e. $\alpha$ is determined by the relation

$$
\alpha-\sin \alpha \cos \alpha=2 d \pi / \rho .
$$

(b)

If the cylinder is depressed a small distance $z$ vertically from equilibrium, the nett upward restoring force is the weight of the extra water displaced or $g \rho .2 a \sin \alpha . l z$, directed oppositely to $z$. This is characteristic of simple harmonic motion and hence the Newtonian equation of motion of the buoy is (upon taking account of the extra factor $1 / 3$ )


$$
8 M \ddot{z} / 3=-2 \rho g l z a \sin \alpha \text { or } \ddot{z}+\frac{3 \rho g \sin \alpha}{4 \pi d a} z=0
$$

and this is the standard sinusoidal oscillator equation (like a simple pendulum). The solution is of the type $z=\sin \left(\omega_{z} t\right)$, with the angular frequency

$$
\omega_{z}=\sqrt{\frac{3 \rho g \sin \alpha}{4 \pi d a}}=\sqrt{\frac{3 g \sin \alpha}{2 a(\alpha-\cos \alpha \sin \alpha)}},
$$

where we have used the relation worked out at the end of the first part.
(c)

Without regard to the torque and only paying heed to vertical forces, if the buoy is swung by some angle so that its weight is supported by the nett pressure of the water outside, the volume of water displaced is the same as in equilibrium. Thus the centre of buoyancy remains at the same distance from the centre of the cylinder. Consequently we deduce that the buoyancy arc is an arc of a circle centred at the middle of the cylinder. In other words, the metacentre $M$ of the swinging motion is just the centre of the cylinder. In fact the question assumes this.
We should also notice that the centre of mass $G$ of the buoy is at the point where the rod touches the cylinder, since the masses of rod and cylinder each equal $M$. Of course the cylinder will experience a nett torque when the rod is inclined to the vertical. To find the period of swing, we first need to determine the moment of inertia of the solid cylinder about the central axis; this is just like a disc about the centre. Thus if $M$ is the cylinder mass


$$
I_{0}=M a^{2} / 2\left(=\int_{0}^{a} r^{2} d m=\int_{0}^{a} r^{2} .2 M r d r / a\right)
$$

The next step is to find the moment of inertia of the rod about its middle,

$$
I_{\text {rod }}=\int_{-a}^{a}(M d x / 2 a) \cdot x^{2}=\left[M x^{3} / 6 a\right]_{-a}^{a}=M a^{2} / 3 .
$$

Finally, use the parallel axis theorem to find the moment of inertia of the buoy (cylinder + rod) about the metacentre $M$,

$$
I_{M}=M a^{2} / 2+\left[M a^{2} / 3+M(2 a)^{2}\right]=29 M a^{2} / 6
$$

(In this part we are neglecting the small horizontal motion of the bentre of mass; the water is the only agent which can supply this force!) When the buoy swings by an angle $\theta$ about equilibrium the restoring torque is $2 M g a \sin \theta \simeq 2 M g a \theta$ for small angles, which represents simple harmonic motion (like simple pendulum). Therefore the Newtonian rotational equation of motion is

$$
I_{M} \ddot{\theta} \simeq-2 M g a \theta, \text { or } \ddot{\theta}+\frac{12 g}{29 a}=0
$$

The solution is a sinusoidal function, $\theta \propto \sin \left(\omega_{\theta} t\right)$, with angular frequency

$$
\omega_{\theta}=\sqrt{12 g / 29 a} .
$$

(d)

The accelerometer measurements give

$$
T_{\theta} / T_{z} \simeq 1.5 \text { or }\left(\omega_{z} / \omega_{\theta}\right)^{2} \simeq 9 / 4 \simeq 2.25 . \text { Hence }
$$

$$
2.25=\frac{3 g \sin \alpha}{2 a(\alpha-\sin \alpha \cos \alpha)} \frac{29 a}{12 g}
$$

producing the (transcendental) equation

$$
\alpha-\sin \alpha \cos \alpha \simeq 1.61 \sin \alpha .
$$

Since 1.61 is not far from 1.57 we have discovered that a physically acceptable solution is $\alpha \simeq \pi / 2$, which was to be shown. (In fact a more accurate solution to the above transcendental equation can be found numerically to be $\alpha=1.591$.) Setting alpha $=\pi / 2$ hereafter, to simplify the algebra, $\omega_{z}^{2}=3 g / \pi a$ and $4 d / \rho=1$ to a good approximation. Since the vertical period is 1.0 sec ,

$$
1.0=\left(2 \pi / \omega_{z}\right)^{2}=4 \pi^{3} a / 3 g
$$

giving the radius $a=3 \times 9.8 / 4 \pi^{3}=.237 \mathrm{~m}$.
We can now work out the mass of the buoy (in SI units),

$$
2 M=2 \pi a^{2} l d=2 \pi a^{2} \cdot a \cdot \rho / 4=\pi a^{3} \rho / 2=\pi \times 500 \times(.237)^{3} \simeq 20.9 \mathrm{~kg} .
$$

## Solutions to Original Theoretical Question 3

(a)

Choose a frame where $z$ is along the normal to the mirror and the light rays define the $x-z$ plane. For convenience, recording the energy-momentum in the four-vector form, $\left(p_{x}, p_{y}, p_{z}, E / c\right)$, the initial photon has

$$
P_{i}=\left(p \sin \theta_{i}, 0, p \cos \theta_{i}, p\right)
$$

where $p=E_{i} / c=h f_{i} / c$.


By the given Lorentz transformation rules, in the moving mirror frame the energy-momentum of the incident photon reads

$$
P_{\text {mirror }}=\left(p \sin \theta_{i}, 0, \frac{p \cos \theta_{i}-u p / c}{\sqrt{1-u^{2} / c^{2}}}, \frac{p-u p \cos \theta_{i} / c}{\sqrt{1-u^{2} / c^{2}}}\right)
$$

Assuming the collision is elastic in that frame, the reflected photon has energy-momentum,

$$
P_{\text {mirror }}^{\prime}=\left(p \sin \theta_{i}, 0, \frac{-p \cos \theta_{i}+u p / c}{\sqrt{1-u^{2} / c^{2}}}, \frac{p-u p \cos \theta_{i} / c}{\sqrt{1-u^{2} / c^{2}}}\right)
$$

Tansforming back to the original frame, we find that the reflected photon has

$$
\begin{aligned}
p_{x r} & =p \sin \theta_{i}, \quad p_{y r}=0 \\
p_{z r} & =\frac{\left(-p \cos \theta_{i}+u p / c\right)+u\left(p-u p \cos \theta_{i} / c\right) / c}{1-u^{2} / c^{2}} \\
E_{r} / c & =\frac{\left(p-u p \cos \theta_{i} / c\right)+u\left(-p \cos \theta_{i}+u p / c\right) / c}{1-u^{2} / c^{2}}
\end{aligned}
$$

Simplifying these expressions, the energy-momentum of the reflected photon in the original frame is

$$
P_{r}=\left(p \sin \theta_{i}, 0, \frac{p\left(-\cos \theta_{i}+2 u / c-u^{2} \cos \theta_{i} / c^{2}\right)}{1-u^{2} / c^{2}}, \frac{p\left(1-2 u \cos \theta_{i} / c+u^{2} / c^{2}\right)}{1-u^{2} / c^{2}}\right)
$$

Hence the angle of reflection $\theta_{r}$ is given by

$$
\tan \theta_{r}=-\frac{p_{x r}}{p_{z} r}=\frac{\sin \theta_{i}\left(1-u^{2} / c^{2}\right)}{\cos \theta_{i}-2 u / c+u^{2} \cos \theta_{i} / c^{2}}=\frac{\tan \theta_{i}\left(1-u^{2} / c^{2}\right)}{1+u^{2} / c^{2}-2 u \sec \theta_{i} / c^{2}}
$$

while the ratio of reflected frequency $f_{r}$ to incident frequency $f_{i}$ is simply the energy ratio,

$$
\frac{f_{r}}{f_{i}}=\frac{E_{r}}{E_{i}}=\frac{1-2 u \cos \theta_{i} / c+u^{2} / c^{2}}{1-u^{2} / c^{2}}
$$

[For future use we may record the changes to first order in $u / c$ :

$$
\begin{aligned}
\tan \theta_{r} & \simeq \tan \theta_{i}\left(1+2 u \sec \theta_{i} / c\right) \\
\tan \left(\theta_{r}-\theta_{i}\right) & =\frac{\tan \theta_{r}-\tan \theta_{i}}{1+\tan \theta_{r} \tan \theta_{i}} \simeq \frac{2 u \tan \theta_{i} \sec \theta_{i} / c}{1+\tan ^{2} \theta_{i}} \simeq \frac{2 u \sin \theta_{i}}{c}
\end{aligned}
$$

Thus, $\theta_{r} \simeq \theta_{i}+2 u \sin \theta_{i} / c$ and $f_{r}=f_{i}\left(1-2 u \cos \theta_{i} / c\right)$.]


Hereafter define $\theta_{i}=\theta$. Provided that $b / \cos \theta<a$ the laser light will reflect off the mirror, so $\cos \theta>b / a$ is needed for photon energy-momentum to be imparted to the mirror. Let us then define a critical angle $\alpha$ via $\cos \alpha=b / a$.
The change in the normal component $\Delta p_{\|}$of the momentum of a single photon is

$$
\begin{gathered}
\Delta L=\frac{\Delta p_{\|} b}{\cos \theta}=\frac{b}{\cos \theta}\left[p \cos \theta-\frac{p\left(-\cos \theta+2 u / c-u^{2} \cos \theta / c^{2}\right)}{1+u^{2} / c^{2}}\right] \\
\Delta L=\frac{b p(2 \cos \theta-2 u / c)}{\cos \theta\left(1+u^{2} / c^{2}\right)}=\frac{2 b p(1-u \sec \theta / c)}{\left(1+u^{2} / c^{2}\right)} \simeq 2 b p(1-u \sec \theta / c) .
\end{gathered}
$$

Since $u \cos \theta=\omega b, \Delta L \simeq 2 b p\left(1-\omega b \sec ^{2} \theta / c\right)$ per photon. Suppose $N$ photons strike every second (and $|\theta|$ is less than the critical angle $\alpha$ ). Then in time $d t$ we have $N d t$ photons. But $d t=d \theta / \omega$, so in this time we have,

$$
d L=N \frac{d \theta}{\omega} \times 2 b p\left(\frac{\omega b}{c} \sec ^{2} \theta\right)
$$

Thus the change in $\Delta L$ per revolution is

$$
\frac{d L}{d n}=2 \times \frac{2 b p N}{\omega} \int_{-a}^{a}\left(1-\omega b \sec ^{2} \theta / c\right) d \theta
$$

where $n$ refers to the number of revolutions. So

$$
\frac{d L}{d n} \simeq \frac{8 b p N}{\omega}\left(\alpha-\frac{\omega b}{c} \tan \alpha\right)=\frac{8 b P}{\omega c}\left(\alpha-\frac{\omega b}{c} \tan \alpha\right)
$$

since each photon has energy $p c$ and laser power equals $P=N p c$.
Clearly $\omega_{b} \ll c$ always, so $d L / d n \simeq 8 b P \alpha / \omega c$; thus

$$
\frac{d L}{d t}=\frac{d L}{d n} \frac{d n}{d t}=\frac{\omega}{2 \pi} \frac{d L}{d n}=\frac{4 b P \alpha}{\pi c} .
$$

(c)

Therefore if $I$ is the moment of inertia of the mirror about its axis of rotation,

$$
I \frac{d \omega}{d t} \simeq \frac{4 b P \alpha}{\pi c}, \text { or } \omega(t) \simeq \frac{4 b P \alpha t}{\pi c I} .
$$

[Some students may derive the rate of change of angular velocity using energy conservation, rather than considering the increase of angular momentum of the mirror: To first order in $v / c, E_{r}=$ $E(1-2 u \cos \theta / c)$, therefore the energy imparted to the mirror is

$$
\Delta E=E-E_{r} \simeq \frac{2 u E \cos \theta}{c}=\frac{2 \omega b E}{c}
$$

In one revolution, the number of photons intersected is

$$
\frac{4 \alpha}{2 \pi} \times n \frac{2 \pi}{\omega}=\frac{4 \alpha n}{\omega}
$$

Therefore the rate of increase of rotational energy $\left(E_{\text {rot }}=I \omega^{2} / 2\right)$ is

$$
\frac{d E_{\mathrm{rot}}}{d t}=\frac{4 \alpha N}{\omega} \frac{2 \omega b E}{c} \frac{d n}{d t}=\frac{8 \alpha b P}{c} \frac{\omega}{2 \pi}=\frac{4 \alpha b P \omega}{\pi c}
$$

Thus $I \omega \cdot d \omega / d t=4 \alpha b P / \pi c$, leading to $\omega(t) \simeq 4 \alpha b P t / \pi c I$, again.]
(d)

To estimate the deflection of the beam, one first needs to work out the moment of inertia of a rectangle of mass $m$ and side $2 a$ about the central axis. This is just like a rod. From basic principles,

$$
I=\int_{-a}^{a} \frac{m d x}{2 a} x^{2}=\left[\frac{m x^{3}}{6 a}\right]_{-a}^{a}=\frac{m a^{2}}{3}=\frac{m b^{2} \sec ^{2} \alpha}{3} .
$$

With the stated geometry, $a=b \sqrt{2}$, or $\alpha=45^{\circ}$, so

$$
\omega \simeq \frac{12 \alpha P t \cos ^{2} \alpha}{\pi m c b} \rightarrow \frac{3 P t}{m c a \sqrt{2}}
$$

At the edge, $u=\omega a=3 P t / m c \sqrt{2}$, and the angle of deviation is

$$
\delta=\frac{2 u \sin \alpha}{c}=\frac{3 P t}{m c^{2}}
$$

[Interestingly, it is determined by the ratio of the energy produced by the laser to the rest-mass energy of the mirror.]
Using the given numbers, and in SI units, the deviation is

$$
\xi \simeq 10^{4} \delta=\frac{10^{4} \times 3 \times 100 \times 24 \times 3600}{10^{-3} \times\left(3 \times 10^{8}\right)^{2}} \simeq 2.9 \mathrm{~mm} .
$$



## Solution to Experimental Question 1

## Preliminary: Calculation of Terminal Velocity

When the cylinder is moving at its terminal velocity, the resultant of the three forces acting on the cylinder, gravity, viscous drag and buoyant force, is zero.

$$
V \rho g-6 \pi \kappa \eta r^{m} v_{T}-V \rho^{\prime} g=0
$$

where $V=2 \pi r^{3}$ is the volume of a cylinder (whose height is $2 r$ ).
This gives

$$
v_{r}=C r^{3-m}\left(\rho-\rho^{\prime}\right)
$$

where

$$
C=\frac{g}{3 \kappa \eta}
$$

## Experiment

## Determination of the exponent $m$

Aluminium cylinders of different diameters are dropped into the glycerine. Fall times between specified marks on the measuring cylinder containing the glycerine are recorded for each cylinder. A preliminary experiment should establish that the cylinders have reached their terminal velocity before detailed results are obtained. The measurements are repeated several times for each cylinder and an average fall time is calculated. Table 1 shows a typical set of data. To find the value of $m$ a graph of $\log$ (fall time) as a function of $\log$ (diameter) is plotted as in figure 1 . The slope of the resulting straight line graph is $3-m$ from which a value of $m$ can be determined. A reasonable value for $m$ is 1.33 with an uncertainty of order $\pm 0.1$. The uncertainty is estimated by the deviation from the line of best fit through the data points obtained by drawing other possible lines.

## Determination of the density of glycerine

Cylinders with the same geometry but different densities are dropped into the glycerine and timed as in the first part of the experiment. Table 2 shows a typical set of results. From equation (2) a linear plot of $1 / t$ as a function of density should yield a straight line with an intercept on the density axis corresponding to the density of glycerine. Figure 2 shows a typical plot. Alternatively the terminal velocities could be calculated and plotted against density which would again lead to the same intercept on the density axis. The uncertainty in the measurement can be estimated by drawing other possible straight lines through the data points and noting the change in the value of the intercept.

| Diameter (mm) | Individual readings (s) |  |  |  |  |  | Mean (s) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | 1.44 | 1.56 | 1.44 | 1.37 | 1.44 | 1.41 | 1.44 |
| 4 | 6.22 | 6.06 | 6.16 | 6.13 | 6.13 | 6.22 | 6.15 |
| 8 | 1.80 | 1.82 | 1.78 | 1.84 | 1.82 | 1.81 | 1.82 |
| 5 | 4.06 | 4.34 | 4.09 | 4.12 | 4.25 | 4.13 | 4.13 |

Table 1: Sample data set


Figure 1: Sample plot

Slope $=-\frac{58.2}{66.2} \div \frac{48.5}{93}=-1.67 \quad \therefore m=3-1.67=1.33$

| Material | Individual readings (s) |  |  |  |  |  | Mean (s) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Ti | 3.00 | 2.91 | 2.97 | 2.91 | 2.84 | 2.75 | 2.91 |
| Cu | 1.25 | 1.25 | 1.28 | 1.25 | 1.22 | 1.22 | 1.25 |
| S Steel | 1.31 | 1.32 | 1.38 | 1.44 | 1.31 | 1.34 | 1.33 |
| Al | 6.03 | 6.09 | 6.09 | 6.16 | 6.06 | 6.06 | 6.08 |

Table 2: Sample data set


Figure 2: Sample plot

$$
\rho^{\prime}=(1.1 \pm 0.2) \times 10^{3} \mathrm{~kg} \cdot \mathrm{~m}^{-3}
$$

## Detailed mark allocation

Section I
Reasonable range of data points with a scatter of $\sim 0.1 \mathrm{~s}$
Check that the cylinders have reached their terminal velocity Visual check, or check referred to
Specific data presented
Labelled log-log graph
Data points for all samples, with a reasonable scatter about a straight line on the log-log graph
Calculation of $(3-m)$ from graph
including estimate of error in determining $m$
Reasonable value of $m, \sim 1.33$
Subtotal
Section 2
Reasonable range of data points
Check that the cylinders have reached their terminal velocity Labelled graph of (falltime) $)^{-1}$ vs. density of cylinder
Data points for all samples, with a reasonable scatter about a straight line on the (falltime) ${ }^{-1}$ vs. density of cylinder graph Calculation of the density of glycerine ( $\rho^{\prime}$ ) from this graph Estimate of uncertainty in $\rho^{\prime}$
Reasonable value of $\rho^{\prime}$. "Correct" value is $1.260 \mathrm{~kg} \cdot \mathrm{~m}^{-3}$ Subtotal

## Solution to Experimental Question 2

## Section 1

i. A typical geometric layout is as shown below.
(a) Maximum distance from ruler to screen is advised to increase the spread of the diffraction pattern.
(b) Note that the grating (ruler) lines are horizontal, so that diffraction is in the vertical direction.

ii. Vis a vis the diffraction phenomenon, $\beta=\left(\frac{y}{1400 \mathrm{~mm}}\right)$

The angle $\beta$ is measured using either a protractor (not recommended) or by measuring the value of the fringe separation on the screen, $y$, for a given order $N$.
If the separation between 20 orders is measured, then $N= \pm 10$ ( $N=0$ is central zero order).
The values of $y$ should be tabulated for $N=10$. If students choose other orders, this is also acceptable.

| $N$ | $\pm 10$ | $\pm 10$ | $\pm 10$ | $\pm 10$ | $\pm 10$ | $\pm 10$ | $\pm 10$ | $\pm 10$ | $\pm 10$ | $\pm 10$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $2 y \mathrm{~mm}$ | 39.0 | 38.5 | 39.5 | 41.0 | 37.5 | 38.0 | 39.0 | 38.0 | 37.0 | 37.5 |
| $y \mathrm{~mm}$ | 19.5 | 19.25 | 19.75 | 20.5 | 18.75 | 19.0 | 19.5 | 19.0 | 18.5 | 18.75 |

Mean Value $=(19.25 \pm 1.25) \mathrm{mm}$
i.e. Mean "spot" distance $=19.25 \mathrm{~mm}$ for order $N=10$.

From observation of the ruler itself, the grating period, $h=(0.50 \pm 0.02) \mathrm{mm}$.
Thus in the relation

$$
\begin{aligned}
N \lambda & = \pm h \sin \beta \\
N & =10 \\
h & =0.5 \mathrm{~mm} \\
\sin \beta \simeq \beta & =\frac{y}{1400 \mathrm{~mm}}=0.01375 \\
10 \lambda & =0.006875 \mathrm{~mm} \\
\lambda & =0.0006875 \mathrm{~mm}
\end{aligned}
$$

Since $\beta$ is small, $\frac{\delta \lambda}{\lambda} \simeq \frac{\delta h}{h}+\frac{\delta y}{y} \simeq 10 \%$
i.e. measured $\lambda=(690 \pm 70) \mathrm{nm}$

The accepted value is 680 nm so that the departure from accepted value equals $1.5 \%$.

## Section 2

This section tests the student's ability to make semi-quantitative measurements and the use of judgement in making observations.
i. Using the $T=50 \%$ transmission disc, students should note that the transmission through the tank is greater than this value. Using a linear approximation, $75 \%$ could well be estimated. Using the hint about the eye's logarithmic response, the transmission through the tank could be estimated to be as high as $85 \%$.
Any figure for transmission between $75 \%$ and $85 \%$ is acceptable.
ii. Calculation of the transmission through the tank, using

$$
T=1-R=1-\left(\frac{n_{1}-n_{2}}{n_{1}+n_{2}}\right)^{2}
$$

for each of the four surfaces of the tank, and assuming $n=1.59$ for the perspex, results in a total transmission

$$
T_{\text {total }}=80.80 \%
$$

## Section 3

With water in the tank, surfaces 2 and 3 become perspex/water interfaces instead of perspex/air interfacs, as in (ii).

The resultant value is

$$
T_{\text {total }}=88.5 \%
$$

## Section 4



Possible configuration for section 4 (and sections 2 and 3)

With pure water in the tank only, we see from Section 3 that the transmission $T$ is

$$
T_{\text {Water }} \simeq 88 \%
$$

The aim here is to determine the beam divergence (scatter) and transmission as a function of milk concentration. Looking down on the tank, one sees

i. The entrance beam diameter is 2.00 mm . The following is an example of the calculations expected: With 0.5 mL milk added to the 50 mL water, we find

$$
\text { Scatterer concentration }=\frac{0.5}{50}=1 \%=0.01
$$

Scattering angle

$$
2 x^{\prime}=2.2 \mathrm{~mm} \quad ; \quad 2 \theta^{\prime}=\frac{2 x^{\prime}}{30}=0.073
$$

Transmission estimated with the assistance of the neutral density filters

$$
T_{\text {total }}=0.7 .
$$

Hence

$$
T_{\mathrm{milk}}=\frac{0.7}{0.88}=0.79
$$

Note that

$$
\begin{equation*}
T_{\text {milk }}=\frac{T_{\text {total }}}{T_{\text {water }}} \quad \text { and } \quad T_{\text {water }}=0.88 \tag{1}
\end{equation*}
$$

If students miss the relationship (1), deduct one mark.
ii. \& iii. One thus obtains the following table of results. $2 \theta^{\prime}$ can be determined as shown above, OR by looking down onto the tank and using the protractor to measure the value of $2 \theta^{\prime}$. It is important to note that even in the presence of scattering, there is still a direct beam being transmitted. It is much stronger than the scattered radiation intensity, and some skill will be required in measuring the scattering angle $2 \theta^{\prime}$ using either method. Making the correct observations requires observational judgement on the part of the student.
Typical results are as follows:

| Milk volume $(\mathrm{mL})$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\%$ Concentration | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $2 x^{\prime}$ | 2.00 | 2.2 | 6.2 | 9.4 | 12 | Protractor |  |  |  |
| $2 \theta^{\prime}$ (Degrees) | $\sim 0$ | 4 | 12 | 18 | 23 | 28 | 36 | 41 | 48 |
| $T_{\text {milk }}$ | 1.0 | 0.79 | 0.45 | 0.22 | 0.15 | 0.12 | 0.08 | 0.06 | 0.05 |

iii. From the graphed results in Figure 1, one obtains an approximately linear relationship between milk concentration, $C$, and scattering angle, $2 \theta^{\prime}(=\phi)$ of the form

$$
\phi=6 C .
$$

iv. Assuming the given relation

$$
I=I_{0} e^{-\mu z}=T_{\text {milk }} I_{0}
$$

where $z$ is the distance into the tank containing milk/water.
We have

$$
T_{\mathrm{milk}}=e^{-\mu z}
$$

Thus

$$
\ln T_{\text {milk }}=-\mu z, \text { and } \mu=\text { constant } \times C
$$

Hence $\ln T_{\text {milk }}=-\alpha z C$.
Since $z$ is a constant in this experiment, a plot of $\ln T_{\text {milk }}$ as a function of $C$ should yield a straight line. Typical data for such a plot are as follows:

| \% Concentration | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{\text {milk }}$ | 1.0 | 0.79 | 0.45 | 0.22 | 0.15 | 0.12 | 0.08 | 0.06 | 0.05 |
| $\ln T_{\text {milk }}$ | 0 | -0.24 | -0.8 | -1.51 | -1.90 | -2.12 | -2.53 | -2.81 | -3.00 |

An approximately linear relationship is obtained, as shown in Figure 2, between $\ln T_{\text {milk }}$ and $C$, the concentration viz.

$$
\ln T_{\mathrm{milk}} \simeq-0.4 C=-\mu z
$$

Thus we can write

$$
T_{\mathrm{milk}}=e^{-0.4 C}=e^{-\mu z}
$$

For the tank used, $z=25 \mathrm{~mm}$ and thus

$$
0.4 C=25 \mu \quad \text { or } \quad \mu=0.016 C \quad \text { whence } \quad \alpha=0.016 \mathrm{~mm}^{-1} \%^{-1}
$$

By extrapolation of the graph of $\ln T_{\text {milk }}$ versus concentration $C$, one finds that for a scatterer concentration of $10 \%$

$$
\mu=0.160 \mathrm{~mm}^{-1} .
$$



Figure 1: Sample plot


Figure 2: Sample plot

## Detailed Mark Allocation

Section 1
A clear diagram illustrating geometry used with appropriate allocations
Optimal geometry used - as per model solution (laser close to ruler)
Multiple measurements made to ascertain errors involved
Correctly tabulated results
Sources of error including suggestion of ruler variation
(suggested by non-ideal diffraction pattern)
Calculation of uncertainty
Final result
Allocated as per:

$$
\begin{aligned}
& \pm 10 \%(612,748 \mathrm{~nm}) \\
& \pm 20 \%(544,816 \mathrm{~nm}) \\
& \pm \text { anything worse }
\end{aligned}
$$

## Section 2

For evidence of practical determination of transmission rather than simply "back calculating". Practical range $70-90 \%$
For correct calculation of transmission
(no more than 3 significant figures stated)

## Section 3

Correct calculation with no more than 3 significant figures stated and an indication that the measurement was performed
Section 4
Illustrative diagram including viewing geometry used, i.e. horizontal/vertical
For recognizing the difference between scattered light and the straight-through beam [1]
For taking the $T_{\text {water }}$ into account when calculating $T_{\text {milk }}$
Correctly calculated and tabulated results of $T_{\text {milk }}$ with results within $20 \%$ of model solution $\quad$ [1]
Using a graphical technique for determining the relationship between scatter angle and milk concentration
Using a graphical technique to extrapolate $T_{\text {milk }}$ to $10 \%$ concentration
Final result for $\mu$
Allocated as $\pm 40 \%$ [2], $\pm 60 \%$ [1], anything worse [0]
A reasonable attempt to consider uncertainties
TOTAL

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PART 4


$$
\begin{aligned}
& z^{2} d \psi=\mathcal{F}^{2} d t \| \quad, c^{2}-q^{2}=\frac{\varphi^{2} c^{4}}{\sigma^{2}} \quad q^{2}=e^{2}\left(1-\frac{\psi^{2}}{\sigma^{2}}\right)=\left(z^{2}\right. \\
& i^{2}\left(1-\frac{\varphi^{2} c^{2}}{b^{2}}\right) \frac{2^{2} d y^{2}}{\mathcal{F}^{2}}=2^{2} d y^{2}+d z^{2}
\end{aligned}
$$

$$
\begin{aligned}
& z_{1}+z_{z}=-\frac{\varphi_{2}^{2} K \mathcal{K}}{\sigma^{2}-\varphi_{2}^{2} c^{2}} \quad \quad \quad, z_{z}=-\frac{\varphi_{2}^{2} \mathcal{K}^{2} \mu^{2}-\mathcal{F}^{2} \xi}{c^{2}\left(\zeta^{2} \emptyset-\varphi_{2}^{2} c^{2}\right)}
\end{aligned}
$$

$$
\begin{aligned}
& 2 \cdot 1_{6}=(2 \pi)^{\prime} \frac{a^{3}}{T} \\
& \text { Example of «Old Masters'» original theoretical work. } \\
& \varepsilon=\varphi_{-} c \bar{c}_{i}
\end{aligned}
$$



# Commission for the Theoretical Competiton: 

Per Chr. Hemmer<br>Alex Hansen<br>Eivind Hiis Hauge<br>Kjell Mork<br>Kåre Olaussen<br>Norwegian University of Science and Technology, Trondheim

$$
\&
$$

Torgeir Engeland<br>Yuri Galperin<br>Anne Holt<br>Asbjørn Kildal<br>Leif Veseth<br>University of Oslo

## 27 ${ }^{\text {th }}$ INTERNATIONAL PHYSICS OLYMPIAD OSLO, NORWAY

## THEORETICAL COMPETITION <br> JULY 21996

## Time available: 5 hours

## READ THIS FIRST :

1. Use only the pen provided
2. Use only the marked side of the paper
3. Each problem should be answered on separate sheets
4. In your answers please use primarily equations and numbers, and as little text as possible
5. Write at the top of every sheet in your report:

- Your candidate number (IPhO identification number)
- The problem number and section identification, e.g. 2/a
- Number each sheet consecutively

6. Write on the front page the total number of sheets in your report


This set of problems consists of 7 pages.

## PROBLEM 1

(The five parts of this problem are unrelated)
a) Five $1 \Omega$ resistances are connected as shown in the figure. The resistance in the conducting wires (fully drawn lines) is negligible.


100 Determine the resulting resistance $R$ between A and B . (1 point)
b)


A skier starts from rest at point A and slides down the hill, without turning or braking. The friction coefficient is $\mu$. When he stops at point B , his horizontal displacement is $s$. What is the height difference $h$ between points A and B? (The velocity of the skier is small so that the additional pressure on the snow due to the curvature can be neglected. Neglect also the friction of air and the dependence of $\mu$ on the velocity of the skier.) (1.5 points)
c) A thermally insulated piece of metal is heated under atmospheric pressure by an electric current so that it receives electric energy at a constant power $P$. This leads to an increase of the absolute temperature $T$ of the metal with time $t$ as follows:

$$
T(t)=T_{0}\left[1+a\left(t-t_{0}\right)\right]^{1 / 4}
$$

Here $a, t_{0}$ and $T_{0}$ are constants. Determine the heat capacity $C_{p}(T)$ of the metal (temperature dependent in the temperature range of the experiment). (2 points)
d) A black plane surface at a constant high temperature $T_{h}$ is parallel to another black plane surface at a constant lower temperature $T_{l}$. Between the plates is vacuum.

In order to reduce the heat flow due to radiation, a heat shield consisting of two thin black plates, thermally isolated from each other, is placed between the warm and the cold surfaces and parallel to these. After some time stationary conditions are obtained.


By what factor $\xi$ is the stationary heat flow reduced due to the presence of the heat shield? Neglect end effects due to the finite size of the surfaces. (1.5 points)
e) Two straight and very long nonmagnetic conductors $C_{+}$and $C_{-}$, insulated from each other, carry a current $I$ in the positive and the negative $z$ direction, respectively. The cross sections of the conductors (hatched in the figure) are limited by circles of diameter $D$ in the $x-y$ plane, with a distance $D / 2$ between the centres. Thereby the resulting cross sections each have an area $\left(\frac{1}{12} \pi+\frac{1}{8} \sqrt{3}\right) D^{2}$. The current in each conductor is uniformly distributed over the cross section.


Determine the magnetic field $B(x, y)$ in the space between the conductors. (4 points)

## PROBLEM 2

The space between a pair of coaxial cylindrical conductors is evacuated. The radius of the inner cylinder is $a$, and the inner radius of the outer cylinder is $b$, as shown in the figure below. The outer cylinder, called the anode, may be given a positive potential $V$ relative to the inner cylinder. A static homogeneous magnetic field $\vec{B}$ parallel to the cylinder axis, directed out of the plane of the figure, is also present. Induced charges in the conductors are neglected.

We study the dynamics of electrons with rest mass $m$ and charge $-e$. The electrons are released at the surface of the inner cylinder.

a) First the potential $V$ is turned on, but $\vec{B}=0$. An electron is set free with negligible velocity at the surface of the inner cylinder. Determine its speed $v$ when it hits the anode. Give the answer both when a non-relativistic treatment is sufficient, and when it is not. (1 point)

For the remaining parts of this problem a non-relativistic treatment suffices.
b) Now $V=0$, but the homogeneous magnetic field $\vec{B}$ is present. An electron starts out with an initial velocity $\vec{v}_{0}$ in the radial direction. For magnetic fields larger than a critical value $B_{c}$, the electron will not reach the anode. Make a sketch of the trajectory of the electron when $B$ is slightly more than $B_{c}$. Determine $B_{c}$. (2 points)

From now on both the potential $V$ and the homogeneous magnetic field $\vec{B}$ are present.
c) The magnetic field will give the electron a non-zero angular momentum $L$ with respect to the cylinder axis. Write down an equation for the rate of change $d L / d t$ of the angular momentum. Show that this equation implies that

$$
L-k e B r^{2}
$$

is constant during the motion, where $k$ is a definite pure number. Here $r$ is the distance from the cylinder axis. Determine the value of $k$. (3 points)
d) Consider an electron, released from the inner cylinder with negligible velocity, that does not reach the anode, but has a maximal distance from the cylinder axis equal to $r_{m}$. Determine the speed $v$ at the point where the radial distance is maximal, in terms of $r_{m}$. (1 point)
e) We are interested in using the magnetic field to regulate the electron current to the anode. For $B$ larger than a critical magnetic field $B_{c}$, an electron, released with negligible velocity, will not reach the anode. Determine $B_{c}$. (l point)
f) If the electrons are set free by heating the inner cylinder an electron will in general have an initial nonzero velocity at the surface of the inner cylinder. The component of the initial velocity parallel to $\vec{B}$ is $v_{B}$, the components orthogonal to $\vec{B}$ are $v_{r}$ (in the radial direction) and $v_{\varphi}$ (in the azimuthal direction, i.e. orthogonal to the radial direction).

Determine for this situation the critical magnetic field $B_{c}$ for reaching the anode. (2 points)

## PROBLEM 3

In this problem we consider some gross features of the magnitude of mid-ocean tides on earth. We simplify the problem by making the following assumptions:
(i) The earth and the moon are considered to be an isolated system,
(ii) the distance between the moon and the earth is assumed to be constant,
(iii) the earth is assumed to be completely covered by an ocean,
(iv) the dynamic effects of the rotation of the earth around its axis are neglected, and
(v) the gravitational attraction of the earth can be determined as if all mass were concentrated at the centre of the earth.

The following data are given:
Mass of the earth: $M=5.98 \cdot 10^{24} \mathrm{~kg}$
Mass of the moon: $M_{m}=7.3 \cdot 10^{22} \mathrm{~kg}$
Radius of the earth: $R=6.37 \cdot 10^{6} \mathrm{~m}$
Distance between centre of the earth and centre of the moon:
$L=3.84 \cdot 10^{8} \mathrm{~m}$
The gravitational constant: $G=6.67 \cdot 10^{-11} \mathrm{~m}^{3} \mathrm{~kg}^{-1} \mathrm{~s}^{-2}$.
a) The moon and the earth rotate with angular velocity $\omega$ about their common centre of mass, $C$. How far is $C$ from the centre of the earth? (Denote this distance by $l$.)

Determine the numerical value of $\omega$. (2 points)

We now use a frame of reference that is co-rotating with the moon and the center of the earth around $C$. In this frame of reference the shape of the liquid surface of the earth is static.


In the plane $P$ through $C$ and orthogonal to the axis of rotation the position of a point mass on the liquid surface of the earth can be described by polar coordinates $r, \varphi$ as shown in the figure. Here $r$ is the distance from the centre of the earth.

We will study the shape

$$
r(\varphi)=R+h(\varphi)
$$

of the liquid surface of the earth in the plane $P$.
b) Consider a mass point (mass $m$ ) on the liquid surface of the earth (in the plane $P$ ). In our frame of reference it is acted upon by a centrifugal force and by gravitational forces from the moon and the earth. Write down an expression for the potential energy corresponding to these three forces.

Note: Any force $F(r)$, radially directed with respect to some origin, is the negative derivative of a spherically symmetric potential energy $V(r)$ :
$F(r)=-V^{\prime}(r)$. (3 points)
c) Find, in terms of the given quantities $M, M_{m}$, etc, the approximate form $h(\varphi)$ of the tidal bulge. What is the difference in meters between high tide and low tide in this model?

You may use the approximate expression

$$
\frac{1}{\sqrt{1+a^{2}-2 a \cos \theta}} \approx 1+a \cos \theta+\frac{1}{2} a^{2}\left(3 \cos ^{2} \theta-1\right),
$$

valid for $a$ much less than unity.
In this analysis make simplifying approximations whenever they are reasonable. (5 points)
$27^{\text {th }}$ INTERNATIONAL PHYSICS OLYMPIAD

# 27 ${ }^{\text {th }}$ INTERNATIONAL PHYSICS OLYMPIAD OSLO, NORWAY 

## THEORETICAL COMPETITION <br> JULY 21996

## Solution Problem 1

a) The system of resistances can be redrawn as shown in the figure:


The equivalent drawing of the circuit shows that the resistance between point c and point A is $0.5 \Omega$, and the same between point d and point B . The resistance between points A and B thus consists of two connections in parallel: the direct $1 \Omega$ connection and a connection consisting of two $0.5 \Omega$ resistances in series, in other words two parallel $1 \Omega$ connections. This yields

$$
R=\underline{\underline{0.5 \Omega}} .
$$

b) For a sufficiently short horizontal displacement $\Delta s$ the path can be considered straight. If the corresponding length of the path element is $\Delta L$, the friction force is given by

$$
\mu m g \frac{\Delta s}{\Delta L}
$$

and the work done by the friction force equals force times displacement:

$$
\mu m g \frac{\Delta s}{\Delta L} \cdot \Delta L=\mu m g \Delta s
$$



Adding up, we find that along the whole path the total work done by friction forces i $\mu m g s$. By energy conservation this must equal the decrease $m g h$ in potential energy of the skier. Hence

$$
h=\underline{\underline{\mu s}} .
$$

c) Let the temperature increase in a small time interval $d t$ be $d T$. During this time interval the metal receives an energy $P d t$.

The heat capacity is the ratio between the energy supplied and the temperature increase:

$$
C_{p}=\frac{P d t}{d T}=\frac{P}{d T / d t} .
$$

The experimental results correspond to

$$
\frac{d T}{d t}=\frac{T_{0}}{4} a\left[1+a\left(t-t_{0}\right)\right]^{-3 / 4}=T_{0} \frac{a}{4}\left(\frac{T_{0}}{T}\right)^{3} .
$$

Hence

$$
C_{p}=\frac{P}{d T / d t}=\frac{4 P}{a T_{0}{ }^{4}} T^{3} .
$$

(Comment: At low, but not extremely low, temperatures heat capacities of metals follow such a $T^{3}$ law.)
d)


Under stationary conditions the net heat flow is the same everywhere:

$$
\begin{aligned}
& J=\sigma\left(T_{h}^{4}-T_{1}^{4}\right) \\
& J=\sigma\left(T_{1}^{4}-T_{2}^{4}\right) \\
& J=\sigma\left(T_{2}^{4}-T_{l}^{4}\right)
\end{aligned}
$$

Adding these three equations we get

$$
3 J=\sigma\left(T_{h}^{4}-T_{l}^{4}\right)=J_{0}
$$

where $J_{0}$ is the heat flow in the absence of the heat shield. Thus $\xi=J / J_{0}$ takes the value

$$
\xi=\underline{\underline{1 / 3}} .
$$

e) The magnetic field can be determined as the superposition of the fields of two cylindrical conductors, since the effects of the currents in the area of intersection cancel. Each of the cylindrical conductors must carry a larger current $I^{\prime}$, determined so that the fraction $I$ of it is carried by the actual cross section (the moon-shaped area). The ratio between the currents $I$ and $I^{\prime}$ equals the ratio between the cross section areas:

$$
\frac{I}{I^{\prime}}=\frac{\left(\frac{\pi}{12}+\frac{\sqrt{3}}{8}\right) D^{2}}{\frac{\pi}{4} D^{2}}=\frac{2 \pi+3 \sqrt{3}}{6 \pi} .
$$

Inside one cylindrical conductor carrying a current $I^{\prime}$ Ampère's law yields at a distance $r$ from the axis an azimuthal field

$$
B_{\phi}=\frac{\mu_{0}}{2 \pi r} \frac{I^{\prime} \pi r^{2}}{\frac{\pi}{4} D^{2}}=\frac{2 \mu_{0} I^{\prime} r}{\pi D^{2}} .
$$

The cartesian components of this are

$$
B_{x}=-B_{\phi} \frac{y}{r}=-\frac{2 \mu_{0} I^{\prime} y}{\pi D^{2}} ; \quad B_{y}=B_{\phi} \frac{x}{r}=\frac{2 \mu_{0} I^{\prime} x}{\pi D^{2}} .
$$

For the superposed fields, the currents are $\pm I^{\prime}$ and the corresponding cylinder axes are located at $x=\mp D / 4$.

The two $x$-components add up to zero, while the $y$-components yield

$$
B_{y}=\frac{2 \mu_{0}}{\pi D^{2}}\left[I^{\prime}(x+D / 4)-I^{\prime}(x-D / 4)\right]=\frac{\mu_{0} I^{\prime}}{\pi D}=\frac{6 \mu_{0} I}{\underline{\underline{(2 \pi+3 \sqrt{3}) D}}},
$$

i.e., a constant field. The direction is along the positive $y$-axis.

## Solution Problem 2

a) The potential energy gain $\mathrm{e} V$ is converted into kinetic energy. Thus

$$
\begin{array}{ll}
\frac{1}{2} m v^{2}=e V & \text { (non-relativistically) } \\
\frac{m c^{2}}{\sqrt{1-2^{2} / c^{2}}}-m c^{2}=e V & \text { (relativistically). }
\end{array}
$$

Hence

$$
v=\left\{\begin{array}{lc}
\sqrt{2 e V / m} & \text { (non- relativistically) }  \tag{1}\\
c \sqrt{1-\left(\frac{m c^{2}}{m c^{2}+e V}\right)^{2}} & \text { (relativistically). }
\end{array}\right.
$$

b) When $V=0$ the electron moves in a homogeneous static magnetic field. The magnetic Lorentz force acts orthogonal to the velocity and the electron will move in a circle. The initial velocity is tangential to the circle.

The radius $R$ of the orbit (the "cyclotron radius") is determined by equating the centripetal force and the Lorentz force:
i.e.

$$
e B v_{0}=\frac{m v_{0}^{2}}{R},
$$

$$
\begin{equation*}
B=\frac{m v_{0}}{e R} . \tag{2}
\end{equation*}
$$



From the figure we see that in the critical case the radius $R$ of the circle satisfies

$$
\sqrt{a^{2}+R^{2}}=b-R
$$

By squaring we obtain
i.e.

$$
a^{2}+R^{2}=b^{2}-2 b R+R^{2},
$$

$$
R=\left(b^{2}-a^{2}\right) / 2 b
$$

Insertion of this value for the radius into the expression (2) gives the critical field

$$
B_{c}=\frac{m v_{0}}{e R}=\frac{2 b m v_{0}}{\left(b^{2}-a^{2}\right) e} .
$$

c) The change in angular momentum with time is produced by a torque. Here the azimuthal component $F_{\phi}$ of the Lorentz force $\vec{F}=(-e) B \times \vec{v}$ provides a torque $F_{\phi} r$. It is only the radial component $v_{r}=d r / d t$ of the velocity that provides an azimuthal Lorentz force. Hence

$$
\frac{d L}{d t}=e B r \frac{d r}{d t},
$$

which can be rewritten as

$$
\frac{d}{d t}\left(L-\frac{e B r^{2}}{2}\right)=0 .
$$

Hence

$$
\begin{equation*}
C=\underline{\underline{L-\frac{1}{2}} e B r^{2}} \tag{3}
\end{equation*}
$$

is constant during the motion. The dimensionless number $k$ in the problem text is thus $k=\underline{1 / 2}$.
d) We evaluate the constant $C$, equation (3), at the surface of the inner cylinder and at the maximal distance $r_{\mathrm{m}}$ :

$$
0-\frac{1}{2} e B a^{2}=m v r_{m}-\frac{1}{2} e B r_{m}^{2}
$$

which gives

$$
\begin{equation*}
v=\frac{\frac{e B\left(r_{m}^{2}-a^{2}\right)}{2 m r_{m}}}{} . \tag{4}
\end{equation*}
$$

Alternative solution: One may first determine the electric potential $V(r)$ as function of the radial distance. In cylindrical geometry the field falls off inversely proportional to $r$, which requires a logarithmic potential, $V(s)=c_{1} \ln r+c_{2}$. When the two constants are determined to yield $V(a)=0$ and $V(b)=V$ we have

$$
V(r)=V \frac{\ln (r / a)}{\ln (b / a)} .
$$

The gain in potential energy, $\operatorname{sV}\left(r_{m}\right)$, is converted into kinetic energy:

$$
\frac{1}{2} m v^{2}=e V \frac{\ln \left(r_{m} / a\right)}{\ln (b / a)} .
$$

Thus

$$
\begin{equation*}
v=\sqrt{\frac{2 e V}{m} \frac{\ln \left(r_{m} / a\right)}{\ln (b / a)}} . \tag{5}
\end{equation*}
$$

(4) and (5) seem to be different answers. This is only apparent since $r_{m}$ is not an independent parameter, but determined by $B$ and $V$ so that the two answers are identical.
e) For the critical magnetic field the maximal distance $r_{m}$ equals $b$, the radius of the outer cylinder, and the speed at the turning point is then

$$
v=\frac{e B\left(b^{2}-a^{2}\right)}{2 m b} .
$$

Since the Lorentz force does no work, the corresponding kinetic energy $\frac{1}{2} m v^{2}$ equals $e V$ (question a):

$$
v=\sqrt{2 e V / m}
$$

The last two equations are consistent when

$$
\frac{e B\left(b^{2}-a^{2}\right)}{2 m b}=\sqrt{2 e V / m}
$$

The critical magnetic field for current cut-off is therefore

$$
B_{c}=\frac{2 b}{\underline{b^{2}-a^{2}} \sqrt{\frac{2 m V}{e}}}
$$

f) The Lorentz force has no component parallel to the magnetic field, and consequently the velocity component $v_{B}$ is constant under the motion. The corresponding displacement parallel to the cylinder axis has no relevance for the question of reaching the anode.

Let $v$ denote the final azimuthal speed of an electron that barely reaches the anode. Conservation of energy implies that

$$
\frac{1}{2} m\left(v_{B}^{2}+v_{\phi}^{2}+v_{r}^{2}\right)+e V=\frac{1}{2} m\left(v_{B}^{2}+v^{2}\right),
$$

giving

$$
\begin{equation*}
v=\sqrt{v_{r}^{2}+v_{\phi}^{2}+2 e V / m} \tag{6}
\end{equation*}
$$

Evaluating the constant $C$ in (3) at both cylinder surfaces for the critical situation we have

$$
m v_{\phi} a-\frac{1}{2} e B_{c} a^{2}=m v b-\frac{1}{2} e B_{c} b^{2} .
$$

Insertion of the value (6) for the velocity $v$ yields the critical field

$$
B_{c}=\frac{2 m\left(v b-v_{\phi} a\right)}{e\left(b^{2}-a^{2}\right)}=\frac{2 m b}{e\left(b^{2}-a^{2}\right)}\left[\sqrt{v_{r}^{2}+v_{\phi}^{2}+2 e V / m}-v_{\phi} a / b\right] .
$$

## Solution Problem 3

a) With the centre of the earth as origin, let the centre of mass $C$ be located at $\vec{l}$. The distance $l$ is determined by

$$
M l=M_{m}(L-l),
$$

which gives

$$
\begin{equation*}
l=\frac{M_{m}}{M+M_{m}} L=\underline{4.63 \cdot 10^{6} \mathrm{~m}}, \tag{1}
\end{equation*}
$$

less than $R$, and thus inside the earth.
The centrifugal force must balance the gravitational attraction between the moon and the earth:

$$
M \omega^{2} l=G \frac{M M_{m}}{L^{2}}
$$

which gives

$$
\begin{equation*}
\omega=\sqrt{\frac{G M_{m}}{L^{2} l}}=\underline{\underline{\frac{G\left(M+M_{m}\right)}{L^{3}}}}=\underline{\underline{2.67 \cdot 10^{-6} \mathrm{~s}^{-1}}} . \tag{2}
\end{equation*}
$$

(This corresponds to a period $2 \pi / \omega=27.2$ days.) We have used (1) to eliminate $l$.
b) The potential energy of the mass point $m$ consists of three contributions:
(1) Potential energy because of rotation (in the rotating frame of reference, see the problem text),

$$
-\frac{1}{2} m \omega^{2} r_{1}^{2}
$$

where $\vec{r}_{1}$ is the distance from $C$. This corresponds to the centrifugal force $m \omega^{2} r_{1}$, directed outwards from $C$.
(2) Gravitational attraction to the earth,

$$
-G \frac{m M}{r} .
$$

(3) Gravitational attraction to the moon,

$$
-G \frac{m M_{m}}{\left|\vec{r}_{m}\right|},
$$

where $\vec{r}_{m}$ is the distance from the moon.
Describing the position of $m$ by polar coordinates $r, \phi$ in the plane orthogonal to the axis of rotation (see figure), we have

$$
\vec{r}_{1}^{2}=(\vec{r}-\vec{l})^{2}=r^{2}-2 r l \cos \phi+l^{2} .
$$



Adding the three potential energy contributions, we obtain

$$
\begin{equation*}
V(\vec{r})=-\frac{1}{2} m \omega^{2}\left(r^{2}-2 r l \cos \phi+l^{2}\right)-G \frac{m M}{r}-G \frac{m M_{m}}{\left|\vec{r}_{m}\right|} . \tag{3}
\end{equation*}
$$

Here $l$ is given by (1) and

$$
\left|\vec{r}_{m}\right|=\sqrt{(\vec{L}-\vec{r})^{2}}=\sqrt{L^{2}-2 \vec{L} \vec{r}+r^{2}}=L \sqrt{1+(r / L)^{2}-2(r / L) \cos \phi} .
$$

c) Since the ratio $r / L=a$ is very small, we may use the expansion

$$
\frac{1}{\sqrt{1+a^{2}-2 a \cos \phi}}=1+a \cos \phi+a^{2} \frac{1}{2}\left(3 \cos ^{2} \phi-1\right)
$$

Insertion into the expression (3) for the potential energy gives

$$
\begin{equation*}
V(r, \phi) / m=-\frac{1}{2} \omega^{2} r^{2}-\frac{G M}{r}-\frac{G M_{m} r^{2}}{2 L^{3}}\left(3 \cos ^{2} \phi-1\right), \tag{4}
\end{equation*}
$$

apart from a constant. We have used that

$$
m \omega^{2} r l \cos \phi-G m M_{m} \frac{r}{L^{2}} \cos \phi=0
$$

when the value of $\omega_{2}$, equation (2), is inserted.

The form of the liquid surface is such that a mass point has the same energy Veverywhere on the surface. (This is equivalent to requiring no net force tangential to the surface.) Putting

$$
r=R+h,
$$

where the tide $h$ is much smaller than R, we have approximately

$$
\frac{1}{r}=\frac{1}{R+h}=\frac{1}{R} \cdot \frac{1}{1+(h / R)} \cong \frac{1}{R}\left(1-\frac{h}{R}\right)=\frac{1}{R}-\frac{h}{R^{2}},
$$

as well as

$$
r^{2}=R^{2}+2 R h+h^{2} \cong R^{2}+2 R h .
$$

Inserting this, and the value (2) of $\omega$ into (4), we have

$$
\begin{equation*}
V(r, \phi) / m=-\frac{G\left(M+M_{m}\right) R}{L^{3}} h+\frac{G M}{R^{2}} h-\frac{G M_{m} r^{2}}{2 L^{3}}\left(3 \cos ^{2} \phi-1\right) \tag{5}
\end{equation*}
$$

again apart from a constant.
The magnitude of the first term on the right-hand side of (5) is a factor

$$
\frac{\left(M+M_{m}\right)}{M}\left(\frac{R}{L}\right)^{3} \cong 10^{-5}
$$

smaller than the second term, thus negligible. If the remaining two terms in equation (5) compensate each other, i.e.,

$$
h=\frac{M_{m} r^{2} R^{2}}{2 M L^{3}}\left(3 \cos ^{2} \phi-1\right),
$$

then the mass point $m$ has the same energy everywhere on the surface. Here $r^{2}$ can safely be approximated by $R^{2}$, giving the tidal bulge

$$
h=\underline{\underline{\frac{M_{m} R^{4}}{2 M L^{3}}}\left(3 \cos ^{2} \phi-1\right) . ~}
$$

The largest value $h_{\max }=M_{m} R^{4} / M L^{3}$ occurs for $\phi=0$ or $\pi$, in the direction of the moon or in the opposite direction, while the smallest value

$$
h_{\min }=-M_{m} R^{4} / 2 M L^{3}
$$

corresponds to $\phi=\pi / 2$ or $3 \pi / 2$.
The difference between high tide and low tide is therefore

$$
h_{\max }-h_{\min }=\frac{3 M_{m} R^{4}}{2 M L^{3}}=\underline{\underline{0.54 \mathrm{~m}}} .
$$

(The values for high and low tide are determined up to an additive constant, but the difference is of course independent of this.)


Here we see the Exam Officer, Michael Peachey (in the middle), with his helper Rod Jory (at the left), both from Australia, as well as the Chief examiner, Per
Chr. Hemmer. The picture was taken in a silent moment during the theory examination. Michael and Rod had a lot of experience from the 1995 IPhO in

Canberra, so their help was very effective and highly appreciated!

## PART 5

## Experimental Competition

Exam commission
Problems in English
The men behind the equipment Model answers in English
Marking form (translated to English)
The last preparations (photos)
Examples of translated texts
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Preparation of the experimental competition was carried out by:


## Commission for the Experimental Competition:

Tom Henning Johansen<br>Børge Holme<br>Svenn L. Andersen<br>Carl Angell<br>Bjørn Berre"<br>Jan Kåre Bording<br>Magne Guttormsen<br>Vidar Hansen<br>Tor Haugset<br>Geir Helgesen ${ }^{\$}$<br>Jan Holtet<br>Randi Haakenaasen<br>Trond Myklebust<br>Jon Samset ${ }^{\S}$<br>from<br>University of Oslo, Oslo<br>§ : Institute for Energy Technology, Kjeller<br>\# : Agricultural University of Norway, Ås

# 27 ${ }^{\text {th }}$ INTERNATIONAL PHYSICS OLYMPIAD OSLO, NORWAY 

## EXPERIMENTAL COMPETITION JULY 41996

## Time available: 5 hours

## READ THIS FIRST :

1. Use only the pen provided.
2. Use only the marked side of the paper.
3. No points will be given for error estimates except in 2c. However, it is expected that the correct number of significant figures are given.
4. When answering problems, use as little text as possible. You get full credit for an answer in the form of a numerical value, a drawing, or a graph with the proper definition of axes, etc.
5. Write on top of every sheet in your report:

- Your candidate number (IPhO ID number)
- The section number
- The number of the sheet

6. Write on the front page the total number of sheets in your report, including graphs, drawings etc.
7. Ensure to include in your report the last page in this set used for answering section 2 a and 3 b , as well as all graphs requested.

SAFETY HAZARD: Be careful with the two vertical blades on the large stand. The blades are sharp!


This set of problems consists of $\mathbf{1 0}$ pages.

## SUMMARY

The set of problems will cover a number of topics in physics. First, some mechanical properties of a physical pendulum will be explored, and you should be able to determine the acceleration of gravity. Then, magnetic forces are added to the pendulum. In this part the magnetic field from a permanent magnet is measured using an electronic sensor. The magnetic moment of a small permanent magnet will be determined. In addition, a question in optics in relation to the experimental setup will be asked.

## INSTRUMENTATION

The following equipment is available (see Figure 1):
A Large aluminium stand
B Threaded brass rod with a tiny magnet in one end (painted white) (iron in the other).
C $\quad 2$ Nuts with a reflecting surface on one side
D Oscillation period timer (clock) with digital display
E Magnetic field (Hall) probe, attached to the large stand
F 9 V battery
G Multimeter, Fluke model 75
H 2 Leads
I Battery connector
J Cylindrical stand made of PVC (grey plastic material)
K Threaded rod with a piece of PVC and a magnet on the top
L Small PVC cylinder of length 25.0 mm (to be used as a spacer)
M Ruler
If you find that the large stand wiggles, try to move it to a different posistion on your table, or use a piece of paper to compensate for the non-flat surface.

The pendulum should be mounted as illustrated in Figure 1. The long threaded rod serves as a physical pendulum, hanging in the large stand by one of the nuts. The groove in the nut should rest on the two vertical blades on the large stand, thus forming a horizontal axis of rotation. The reflecting side of the nut is used in the oscillation period measurement, and should always face toward the timer.

The timer displays the period of the pendulum in seconds with an uncertainty of $\pm 1 \mathrm{~ms}$. The timer has a small infrared light source on the right-hand side of the display (when viewed from the front), and an infrared detector mounted
close to the emitter. Infrared light from the emitter is reflected by the mirror side of the nut. The decimal point lights up when the reflected light hits the detector. For proper detection the timer can be adjusted vertically by a screw (see N in Figure 1). Depending on the adjustment, the decimal point will blink either once or twice each oscillation period. When it blinks twice, the display shows the period of oscillation, $T$. When it blinks once, the displayed number is $2 T$. Another red dot appearing after the last digit indicates low battery. If battery needs to be replaced, ask for assistance.

The multimeter should be used as follows:
Use the "V $\Omega$ " and the "COM" inlets. Turn the switch to the DC voltage setting. The display then shows the DC voltage in volts. The uncertainty in the instrument for this setting is $\pm(0.4 \%+1$ digit $)$.


Figure 1. The instrumentation used.

## THE PHYSICAL PENDULUM

A physical pendulum is an extended physical object of arbitrary shape that can rotate about a fixed axis. For a physical pendulum of mass $M$ oscillating about a horizontal axis a distance, $l$, from the centre of mass, the period, $T$, for small angle oscillations is

$$
\begin{equation*}
T=\frac{2 \pi}{\sqrt{g}} \sqrt{\frac{I}{M l}+l} \tag{1}
\end{equation*}
$$

Here $g$ is the acceleration of gravity, and $I$ is the moment of inertia of the pendulum about an axis parallel to the rotation axis but through the centre of mass.

Figure 2 shows a schematic drawing of the physical pendulum you will be using. The pendulum consists of a cylindrical metal rod, actually a long screw, having length $L$, average radius $R$, and at least one nut. The values of various dimensions and masses are summarised in Table 1. By turning the nut you can place it at any position along the rod. Figure 2 defines two distances, $x$ and $l$, that describe the position of the rotation axis relative to the end of the rod and the centre of mass, respectively.


Figure 2: Schematic drawing of the pendulum with definition of important quantities.

| Length | $L$ | $(400.0 \pm 0.4) \mathrm{mm}$ |
| :--- | :--- | :--- |
| Average radius | $R$ | $(4.4 \pm 0.1) \mathrm{mm}$ |
| Mass | $M_{\text {ROD }}$ | $(210.2 \pm 0.2) \cdot 10^{-3} \mathrm{~kg}$ |
| Distance between screw threads |  | $(1.5000 \pm 0.0008) \mathrm{mm}$ |

Nut

| Height | $h$ | $(9.50 \pm 0.05) \mathrm{mm}$ |
| :--- | :--- | :--- |
| Depth of groove | $d$ | $(0.55 \pm 0.05) \mathrm{mm}$ |
| Mass | $M_{\text {NUT }}$ | $(4.89 \pm 0.03) \cdot 10^{-3} \mathrm{~kg}$ |

Table 1: Dimensions and weights of the pendulum
A reminder from the front page: No points will be given for error estimates except in 2c. However, it is expected that the correct number of significant figures are given.

## Section 1 : Period of oscillation versus rotation axis position (4 marks)

a) Measure the oscillation period, $T$, as a function of the position $x$, and present the results in a table.
b) Plot $T$ as a function of $x$ in a graph. Let 1 mm in the graph correspond to 1 mm in $x$ and 1 ms in $T$. How many positions give an oscillation period equal to $T=950 \mathrm{~ms}, T=1000 \mathrm{~ms}$ and $T=1100 \mathrm{~ms}$, respectively?
c) Determine the $x$ and $l$ value that correspond to the minimum value in $T$.

## Section 2 : Determination of $\boldsymbol{g}$ (5 marks)

For a physical pendulum with a fixed moment of inertia, $I$, a given period, $T$, may in some cases be obtained for two different positions of the rotation axis. Let the corresponding distances between the rotation axis and the centre of mass be $l_{1}$ and $l_{2}$. Then the following equation is valid:

$$
\begin{equation*}
l_{1} l_{2}=\frac{I}{M} \tag{2}
\end{equation*}
$$

a) Figure 6 on the last page in this set illustrates a physical pendulum with an axis of rotation displaced a distance $l_{1}$ from the centre of mass. Use the information given in the figure caption to indicate all positions where a rotation axis parallel to the drawn axis can be placed without changing the oscillation period.
b) Obtain the local Oslo value for the acceleration of gravity $g$ as accurately as possible. Hint: There are more than one way of doing this. New measurements might be necessary. Indicate clearly by equations, drawings, calculations etc. the method you used.
c) Estimate the uncertainty in your measurements and give the value of $g$ with error margins.

## Section 3 : Geometry of the optical timer (3 marks)

a) Use direct observation and reasoning to characterise, qualitatively as well as quantitatively, the shape of the reflecting surface of the nut (the mirror). (You may use the light from the light bulb in front of you).

Options (several may apply):

1. Plane mirror
2. Spherical mirror
3. Cylindrical mirror
4. Cocave mirror
5. Convex mirror

In case of 2-5: Determine the radius of curvature.
b) Consider the light source to be a point source, and the detector a simple photoelectric device. Make an illustration of how the light from the emitter is reflected by the mirror on the nut in the experimantal setup (side view and top view). Figure 7 on the last page in this set shows a vertical plane through the timer display (front view). Indicate in this figure the whole region where the reflected light hits this plane when the pendulum is vertical.

## Section 4 : Measurement of magnetic field (4 marks)

You will now use an electronic sensor (Hall-effect sensor) to measure magnetic field. The device gives a voltage which depends linearly on the vertical field through the sensor. The field-voltage coefficient is $\Delta V / \Delta B=22.6 \mathrm{~V} / \mathrm{T}$ (Volt/ Tesla). As a consequence of its design the sensor gives a non-zero voltage (zero-offset voltage) in zero magnetic field. Neglect the earth's magnetic field.


Figure 3: Schematics of the magnetic field detector system
a) Connect the sensor to the battery and voltmeter as shown above. Measure the zero-offset voltage, $V_{0}$.

A permanent magnet shaped as a circular disk is mounted on a separate stand. The permanent magnet can be displaced vertically by rotating the mount screw, which is threaded identically to the pendulum rod. The dimensions of the permanent magnet are; thickness $t=2.7 \mathrm{~mm}$, radius $r=12.5 \mathrm{~mm}$.
b) Use the Hall sensor to measure the vertical magnetic field, $B$, from the permanent magnet along the cylinder axis, see Figure 4. Let the measurements cover the distance from $y=26 \mathrm{~mm}$ (use the spacer) to $y=3.5 \mathrm{~mm}$, where $y=1 \mathrm{~mm}$ corresponds to the sensor and permanent magnet being in direct contact. Make a graph of your data for $B$ versus $y$.


Figure 4: Definition of the distance y between top of magnet and the active part of the sensor.
c) It can be shown that the field along the axis of a cylindrical magnet is given by the formula

$$
\begin{equation*}
B(y)=B_{0}\left[\frac{y+t}{\sqrt{(y+t)^{2}+r^{2}}}-\frac{y}{\sqrt{y^{2}+r^{2}}}\right] \tag{3}
\end{equation*}
$$

where $t$ is the cylinder thickness and $r$ is the radius. The parameter $B_{0}$ characterizes the strength of the magnet. Find the value of $B_{0}$ for your permanent magnet. ${ }^{\S}$ Base your determination on two measured $B$-values obtained at different $y$.

## Section 5 : Determination of magnetic dipole moment (4 marks)

A tiny magnet is attached to the white end of the pendulum rod. Mount the pendulum on the stand with its magnetic end down and with $\boldsymbol{x}=\mathbf{1 0 0} \mathbf{~ m m}$. Place the permanent magnet mount under the pendulum so that both the permanent magnet and the pendulum have common cylinder axis. The alignment should be done with the permanent magnet in its lowest position in the mount. (Always avoid close contact between the permanent magnet and the magnetic end of the pendulum.)
a) Let $z$ denote the air gap spacing between the permanent magnet and the lower end of the pendulum. Measure the oscillation period, $T$, as function of the distance, $z$. The measurement series should cover the interval from $z=25 \mathrm{~mm}$ to $z=5.5 \mathrm{~mm}$ while you use as small oscillation amplitude as possible. Be aware of the possibility that the period timer might display $2 T$ (see remark regarding the timer under Instrumentation above). Plot the observed $T$ versus $z$.
b) With the additional magnetic interaction the pendulum has a period of oscillation, $T$, which varies with $z$ according to the relation

$$
\begin{equation*}
\frac{1}{T^{2}} \propto 1+\frac{\mu B_{0}}{M g l} f(z) \tag{4}
\end{equation*}
$$

Here $\propto$ stand for "proportional to", and $\mu$ is the magnetic dipole moment of the tiny magnet attached to the pendulum, and is the parameter determined in section 4c. The function $f(z)$ includes the variation in magnetic field with distance. In Figure 5 on the next page you find the particular $f(z)$ for our experiment, presented as a graph.
Select an appropirate point on the graph to determine the unknown magnetic moment $\mu$.

[^0]

Figure 5. Graph of the dimension-less function $f(z)$ used in section $5 b$.

| Candidate: | Question: | Page ..... of ..... |
| :--- | :--- | :--- |

Axis of rotation


Figure 6. For use in section 2a. Mark all positions where a rotation axis (orthogonal to the plane of the paper) can be placed without changing the oscillation period. Assume for this pendulum (drawn on scale, 1:1) that $I / M=2100 \mathrm{~mm}^{2}$. (Note: In this booklet the size of this figure is about 75\% of the size in the original examination paper.)


Figure 7. For use in section 3b. Indicate the whole area where the reflected light hits when the pendulum is vertical.

## Include this page in your report!

## The men behind the equipment

The equipment for the practical competition was constructed and manufactured at the Mechanics Workshop at the Department of Physics, University of Oslo (see picture below, from left to right: Tor Enger (head of the Mechanics Workshop), Pål Sundbye, Helge Michaelsen, Steinar Skaug Nilsen, and Arvid Andreassen).


The electronic timer was designed and manufactured by Efim Brondz, Department of Physics, University of Oslo (see picture below). About 40.000 soldering points were completed manually, enabling the time-recording during the exam to be smooth and accurate.


Photo: Geir Holm

| Candidate: IPhO ID | Question: 1 | Page 1 of 11 |
| :--- | :--- | :--- |

## 27 ${ }^{\text {th }}$ INTERNATIONAL PHYSICS OLYMPIAD OSLO, NORWAY

Model Answer<br>for the<br>EXPERIMENTAL COMPETITION<br>JULY 41996

These model answers indicate what is required from the candidates to get the maximum score of 20 marks. Some times we have used slightly more text than required; paragraphs written in intuition and a thorough understanding of the physics involved.

Alternative solutions regarded as less elegant or more time consuming are printed in frames like this with white background.

Anticipated INCORRECT answers are printed on grey background and are included to point out places where the students may make mistakes or approximations without being aware of them.

## Section 1:

1a) Threads are $1.50 \mathrm{~mm} / \mathrm{turn}$. Counted turns to measure position $x$.

| Turn no. | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $x[\mathrm{~mm}]$ | 10.0 | 25.0 | 40.0 | 55.0 | 70.0 | 85.0 | 100.0 | 115.0 | 130.0 | 145.0 | 160.0 |
| $T[\mathrm{~ms}]$ | 1023 | 1005 | 989 | 976 | 967 | 964 | 969 | 987 | 1024 | 1094 | 1227 |
| Turn no. | 110 | 120 |  | 46 | 48 | 52 | 54 |  |  |  |  |
| $x[\mathrm{~mm}]$ | 175.0 | 190.0 | 79.0 | 82.0 | 88.0 | 91.0 |  |  |  |  |  |
| $T[\mathrm{~ms}]$ | 1490 | 2303 |  | 964 | 964 | 964 | 965 |  |  |  |  |


| Candidate: IPhO ID | Question: 1 | Page 2 of 11 |
| :--- | :--- | :--- |

1b) Graph: $T(x)$, shown above.

$$
\begin{array}{ll}
T=950 \mathrm{~ms}: & \text { NO positions } \\
T=1000 \mathrm{~ms}: & 2 \text { positions } \\
T=1100 \mathrm{~ms}: & 1 \text { position }
\end{array}
$$

If the answer is given as corresponding $x$-values, and these reflect the number of positions asked for, this answer will also be accepted.

| Candidate: IPhO ID | Question: $1+2$ | Page 3 of 11 |
| :--- | :--- | :--- |

1c) Minimum on graph: $x=84 \mathrm{~mm}$, (estimated uncertainty 1 mm )
By balancing the pendulum horizontally: $l=112.3 \mathrm{~mm}+0.55 \mathrm{~mm}=113 \mathrm{~mm}$


## ALTERNATIVE 1c-1:

$x_{C M}=\frac{M_{R O D} L-M_{N U T} h}{2 M}+\frac{M_{N U T}}{M} x=197.3 \mathrm{~mm}$ for $x=84 \mathrm{~mm}$
gives $l=197.3 \mathrm{~mm}-84 \mathrm{~mm}=113 \mathrm{~mm}$
$M=M_{R O D}+M_{N U T}, h=8.40 \mathrm{~mm}=$ height of nut minus two grooves.

INCORRECT 1c-1: Assuming that the centre of mass for the pendulum coincides with the midpoint, $L / 2$, of the rod gives $l=L / 2-x=116 \mathrm{~mm}$.
(The exact position of the minimum on the graph is $x=84.4 \mathrm{~mm}$. with $l=112.8 \mathrm{~mm}$ )

## Section 2:

2a) $l_{2}=\frac{I}{M l_{1}}=\frac{2100 \mathrm{~mm}^{2}}{60 \mathrm{~mm}}=35 \mathrm{~mm}$

|  |  |  |  |
| :---: | :---: | :---: | :---: |
| Candidate: IPhO ID | Question: 2 | Page 4 of 11 |  |



Figure 6. For use in section 2a. Mark all positions where a rotation axis (orthogonal to the plane of the paper) can be placed without changing the oscillation period. Assume for this pendulum (drawn on scale, 1:1) that $I / M=2100 \mathrm{~mm}^{2}$. (Note: In this booklet the size of this figure is about $75 \%$ of the size in the original examination paper.)


TOP VIEW


Paper plane coincides with the plane of the display

Figure 7. For use in section 3b. Indicate the whole area where the reflected light hits when the pendulum is vertical.

Include this page in your report!

| Candidate: IPhO ID | Question: 2 | Page 5 of 11 |
| :--- | :--- | :--- |

2b) Simple method with small uncertainty: Inverted pendulum.
Equation (1) $+(2) \Rightarrow T_{1}=T_{2}=\frac{2 \pi}{\sqrt{g}} \sqrt{l_{1}+l_{2}} \Leftrightarrow g=\frac{4 \pi^{2}}{T_{1}^{2}}\left(l_{1}+l_{2}\right)$
NOTE: Independent of I/M !
Used both nuts with one nut at the end to maximise $l_{1}+l_{2}$. Alternately adjusted nut positions until equal periods $T_{1}=T_{2}$ :

$T_{1}=T_{2}=1024 \mathrm{~ms}$.
Adding the depth of the two grooves to the measured distance between nuts:
$l_{1}+l_{2}=(259.6+2 \cdot 0.55) \mathrm{mm}=0.2607 \mathrm{~m}$
$g=\frac{4 \pi^{2}}{T_{1}^{2}}\left(l_{1}+l_{2}\right)=\frac{4 \cdot 3.1416^{2} \cdot 0.2607 \mathrm{~m}}{(1.024 \mathrm{~s})^{2}}=\underline{\underline{9.815 \mathrm{~m} / \mathrm{s}^{2}}}$

ALTERNATIVE 2b-1: Finding I(x). Correct but time consuming.
It is possible to derive an expression for I as a function of $x$. By making sensible approximations, this gives:

$$
\frac{I(x)}{M}=\left[\frac{L^{2}}{12}+\frac{M_{N U T}}{M}\left(\frac{L+h}{2}-x\right)^{2}\right] \frac{M_{R O D}}{M}
$$

which is accurate to within $0.03 \%$. Using the correct expression for 1 as a function of $x$ :

$$
l(x)=x_{C M}-x=\frac{M_{R O D} L-M_{N U T} h}{2 M}-\frac{M_{R O D}}{M} x=195.3 \mathrm{~mm}-0.9773 x
$$

equation (1) can be used on any point ( $x, T$ ) to find $g$. Choosing the point ( $85 \mathrm{~mm}, 964 \mathrm{~ms}$ ) gives:
$g=\frac{4 \pi^{2}}{T^{2}}\left[\frac{I(x)}{M \cdot l(x)}+l(x)\right]=\frac{4 \cdot 3.1416^{2} \cdot 0.2311 \mathrm{~m}}{(0.964 \mathrm{~s})^{2}}=\underline{\underline{9.818 \mathrm{~m} / \mathrm{s}^{2}}}$

| Candidate: IPhO ID | Question: 2 | Page 6 of 11 |
| :--- | :--- | :--- |

Using the minimum point on the graph in the way shown below is wrong, since the curve in 1b) , $T(x)=\frac{2 \pi}{\sqrt{g}} \sqrt{\frac{I(x)}{M \cdot l(x)}+l(x)}$ with $I(x) / M$ and $l(x)$ given above, describes a continuum of different pendulums with changing $I(x)$ and moving centre of mass.
Equation (1): $T=\frac{2 \pi}{\sqrt{g}} \sqrt{\frac{I}{M l}+l}$ describes one pendulum with fixed $I$, and does not apply to the curve in 1b).

INCORRECT 2b-1: At the minimum point we have from Equation (2) and 1c):
$l_{1}=l_{2}=l=\sqrt{I / M}=(113 \pm 1) \mathrm{mm}$ Equation (1) becomes
$T_{\min }=\frac{2 \pi}{\sqrt{g}} \sqrt{\frac{l^{2}}{l}+l}=\frac{2 \pi}{\sqrt{g}} \sqrt{2 l}$ and
$g=\frac{8 \pi^{2} l}{T_{\min }{ }^{2}}=\frac{8 \cdot 3.1416^{2} \cdot 0.113 \mathrm{~m}}{(0.964 \mathrm{~s})^{2}}=9.60 \mathrm{~m} / \mathrm{s}^{2}$
Another source of error which may accidentally give a reasonable value is using the wrong value $\mathrm{l}=(116 \pm 1) \mathrm{mm}$ from «INCORRECT 1c-1»:
INCORRECT 2b-2: $g=\frac{8 \pi^{2} l}{T_{\text {min }}{ }^{2}}=\frac{8 \cdot 3.1416^{2} \cdot 0.116 \mathrm{~m}}{(0.964 \mathrm{~s})^{2}}=9.86 \mathrm{~m} / \mathrm{s}^{2}$

Totally neglecting the mass of the nut but remembering the expression for the moment of inertia for a thin rod about a perpendicular axis through the centre of mass, $I=M L^{2} / 12$, gives from equation (2) for the minimum point: $l^{2}=I / M=L^{2} / 12=0.01333 \mathrm{~m}^{2}$. This value is accidentally only $0.15 \%$ smaller than the correct value for $I(x) / M$ at the minimum point on the curve in $\mathbf{1 b}$ ):

$$
\frac{I(x=84.43 \mathrm{~mm})}{M}=\left[\frac{L^{2}}{12}+\frac{M_{N U T}}{M}\left(\frac{L+h}{2}-x\right)^{2}\right] \frac{M_{R O D}}{M}=0.01335 \mathrm{~m}^{2}
$$

| Candidate: IPhO ID | Question: 2 | Page 7 of 11 |
| :--- | :--- | :--- |


(cont.)
Neglecting the term $\frac{M_{N U T}}{M}\left(\frac{L+h}{2}-84.43 \mathrm{~mm}\right)^{2}=0.00033 \mathrm{~m}^{2}$ is nearly compensated by omitting the factor $\frac{M_{R O D}}{M}=0.977$. However, each of these approximations are of the order of $2.5 \%$, well above the accuracy that can be achieved.

INCORRECT 2b-3: At the minimum point equation (2) gives $l^{2}=\frac{I}{M}=\frac{L^{2}}{12}$. Then

$$
\begin{aligned}
& T_{\min }=\frac{2 \pi}{\sqrt{g}} \sqrt{2 l}=\frac{2 \pi}{\sqrt{g}} \sqrt{\frac{2 L}{\sqrt{12}}}=\frac{2 \pi}{\sqrt{g}} \sqrt{\frac{L}{\sqrt{3}}} \text { and } \\
& g=\frac{4 \pi^{2} L}{\sqrt{3} T_{\min }^{2}}=\frac{4 \cdot 3.1416^{2} \cdot 0.4000 \mathrm{~m}}{1.7321 \cdot(0.964 \mathrm{~s})^{2}}=9.81 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

2c) Estimating uncertainty in the logarithmic expression for $g$ :
Let $S \equiv l_{1}+l_{2} \Rightarrow g=\frac{4 \pi^{2} S}{T^{2}}$
$\Delta S=0.3 \mathrm{~mm} \quad \Delta T=1 \mathrm{~ms}$

$$
\begin{aligned}
\frac{\Delta g}{g} & =\sqrt{\left(\frac{\Delta S}{S}\right)^{2}+\left(-2 \frac{\Delta T}{T}\right)^{2}}=\sqrt{\left(\frac{0.3 \mathrm{~mm}}{260.7 \mathrm{~mm}}\right)^{2}+\left(2 \cdot \frac{1 \mathrm{~ms}}{1024 \mathrm{~ms}}\right)^{2}} \\
& =\sqrt{(0.0012)^{2}+(0.0020)^{2}}=0.0023=0.23 \%
\end{aligned}
$$

$\Delta g=0.0023 \cdot 9.815 \mathrm{~m} / \mathrm{s}^{2}=0.022 \mathrm{~m} / \mathrm{s}^{2}$
$\underline{g=(9.82 \pm 0.02) \mathrm{m} / \mathrm{s}^{2}}$

The incorrect methods INCORRECT $2 \mathrm{~b}-1,2 \mathrm{~b}-2$ and $2 \mathrm{~b}-3$ have a similar expressions for g as above. With $\Delta l=1 \mathrm{~mm}$ in INCORRECT $2 \mathrm{~b}-1$ and $2 \mathrm{~b}-2$ we get $\Delta g=0.09 \mathrm{~m} / \mathrm{s}^{2}$.
INCORRECT $2 \mathrm{~b}-3$ should have $\Delta l=0.3 \mathrm{~mm}$ and $\Delta g=0.02 \mathrm{~m} / \mathrm{s}^{2}$.

| Candidate: IPhO ID | Question: $2+3+4$ | Page 8 of 11 |
| :--- | :--- | :--- |

ALTERNATIVE 3 has a very complicated $x$ dependence in $g$. Instead of differentiating $g(x)$ it is easier to insert the two values $x+\Delta x$ and $x-\Delta x$ in the expression in brackets [ ], thus finding an estimate for $\Delta[$ ] and then using the same formula as above.
(The official local value for $g$, measured in the basement of the adjacent building to where the practical exam was held is $g=9.8190178 \mathrm{~m} / \mathrm{s}^{2}$ with uncertainty in the last digit.)

## Section 3.

3a) 3. Cylindrical mirror
4. Concave mirror

Radius of curvature of cylinder, $r=145 \mathrm{~mm}$. (Uncertainty approx. $\pm 5 \mathrm{~mm}$, not asked for.)
(In this set-up the emitter and detector are placed at the cylinder axis. The radius of curvature is then the distance between the emitter/detector and the mirror. )

3b) Three drawings, see Figure 7 on page 4 in this Model Answers.
(The key to understanding this set-up is that for a concave cylindrical mirror with a point source at the cylinder axis, the reflected light will be focused back onto the cylinder axis as a line segment of length twice the width of the mirror.)

## Section 4.

4a) $\mathrm{V}_{\mathrm{O}}=2.464 \mathrm{~V}$ (This value may be different for each set-up.)

4b) Threads are $1.50 \mathrm{~mm} /$ turn. Measured $V(y)$ for each turn. Calculated

$$
B(y)=\left[V(y)-V_{0}\right] \frac{\Delta B}{\Delta V}=\left[V(y)-V_{0}\right] / \frac{\Delta V}{\Delta B} . \quad \text { (Table not requested) }
$$

See graph on next page.

| Candidate: IPhO ID | Question: $4+5$ | Page 9 of 11 |
| :--- | :--- | :--- |

Graph: B(y):


4c)

$$
B_{0}=B(y)\left[\frac{y+t}{\sqrt{(y+t)^{2}+r^{2}}}-\frac{y}{\sqrt{y^{2}+r^{2}}}\right]^{-1}
$$

The point ( $11 \mathrm{~mm}, 48.5 \mathrm{mT}$ ) gives $B_{0}=0.621 \mathrm{~T}$ and $(20 \mathrm{~mm}, 16,8 \mathrm{mT})$ gives $B_{0}=0.601 \mathrm{~T}$. Mean value: $B_{0}=0.61 \mathrm{~T}$ (This value may vary for different magnets.)

## Section 5:

5a) Used the spacer and measured $T(z)$ from $z=25 \mathrm{~mm}$ to 5.5 mm . (Table is not requested.) See plot on next page.

| Candidate: IPhO ID | Question: 5 | Page 10 of 11 |
| :--- | :--- | :--- |

## Graph: T(z):



5b) $l(x=100 \mathrm{~mm})=97.6 \mathrm{~mm}$ (by balancing the pendulum or by calculation as in 1 c ).

$$
M=M_{R O D}+M_{N U T}
$$

Proportionality means: $\frac{1}{T^{2}}=a\left[1+\frac{\mu B_{0}}{M g l} f(z)\right]$ where $a$ is a proportionality constant. Setting $B_{0}=0$ corresponds to having an infinitely weak magnet or no magnet at all. Removing the
large magnet gives: $\mathrm{T}_{0}=968 \mathrm{~ms}$ and $\frac{1}{T_{0}{ }^{2}}=a\left[1+0 \cdot \frac{\mu}{M g l} f(z)\right]$ or $a=\frac{1}{T_{0}{ }^{2}}$.
Selecting the point where $f(z)$, see Fig. 5, changes the least with $z$, i.e., at the maximum, one has $f_{\max }=56.3$. This point must correspond to the minimum oscillation period, which is measured to be $T_{\text {min }}=576 \mathrm{~ms}$.

We will often need the factor

$$
\frac{M g l}{B_{0}}=\frac{0.215 \mathrm{~kg} \cdot 9.82 \mathrm{~m} / \mathrm{s}^{2} \cdot 0.0976 \mathrm{~m}}{0.61 \mathrm{~T}}=0.338 \mathrm{Am}^{2}
$$

The magnetic moment then becomes

$$
\mu=\frac{M g l}{B_{0}} \frac{1}{f_{\max }}\left[\left(\frac{T_{0}}{T}\right)^{2}-1\right]=\frac{0.338 \mathrm{Am}^{2}}{56.3}\left[\left(\frac{968}{576}\right)^{2}-1\right]=\underline{\underline{1.1 \cdot 10^{-2} \mathrm{Am}^{2}}}
$$

ALTERNATIVE 5b-1: Not what is asked for: Using two points to eliminate the proportionality constant $a$ : Equation (4) or $\frac{1}{T^{2}}=a\left[1+\frac{\mu B_{0}}{M g l} f(z)\right]$ gives:

$$
a T_{1}^{2}\left[1+\frac{\mu B_{0}}{M g l} f\left(z_{1}\right)\right]=a T_{2}^{2}\left[1+\frac{\mu B_{0}}{M g l} f\left(z_{2}\right)\right]
$$

$$
\begin{aligned}
& T_{1}^{2}+T_{1}^{2} \frac{\mu B_{0}}{M g l} f\left(z_{1}\right)=T_{2}^{2}+T_{2}^{2} \frac{\mu B_{0}}{M g l} f\left(z_{2}\right) \\
& \frac{\mu B_{0}}{M g l}\left[T_{1}^{2} f\left(z_{1}\right)-T_{2}^{2} f\left(z_{2}\right)\right]=T_{2}^{2}-T_{1}^{2} \\
& \mu=\frac{M g l}{B_{0}} \cdot \frac{T_{2}^{2}-T_{1}^{2}}{T_{1}^{2} f\left(z_{1}\right)-T_{2}^{2} f\left(z_{2}\right)}
\end{aligned}
$$

Choosing two points ( $z_{1}=7 \mathrm{~mm}, T_{1}=580.5 \mathrm{~ms}$ ) and ( $\mathrm{z}_{2}=22 \mathrm{~mm}, T_{2}=841 \mathrm{~ms}$ ). Reading from the graph $f\left(z_{1}\right)=56.0$ and $f\left(z_{2}\right)=12.0$ we get

$$
\mu=0.338 \mathrm{Am}^{2} \cdot \frac{841^{2}-580^{2}}{580^{2} \cdot 56.0-841^{2} \cdot 12.0}=1.2 \cdot 10^{-2} \mathrm{Am}^{2}
$$

| Candidate: | Total score: $++++=$ |
| :--- | :--- |
| Country: | Marker's name: |
| Language: | Comment: |

## Marking Form

## for the Experimental Competition at the 27th International Physics Olympiad Oslo, Norway <br> July 4, 1996

To the marker: Carefully read through the candidate's exam papers and compare with the model answer. You may use the transparencies (handed out) when checking the graph in 1b) and the drawing in 2a). When encountering words or sentences that require translation, postpone marking of this part until you have consulted the interpreter.

Use the table below and mark a circle around the point values to be subtracted. Add vertically the points for each subsection and calculate the score.
NB: Give score 0 if the value comes out negative for any subsection.
Add the scores for each subsection and write the sum in the 'Total score'- box at the upper right. Keep decimals all the way.

If you have questions, consult the marking leader. Good luck, and remember that you will have to defend your marking in front of the team leaders.
(Note: The terms "INCORRECT $2 \mathrm{~b}-1$ " found in the table for subsection 2 c ) and similar terms elsewhere, refer to the Model Answer, in which anticipated incorrect answers were included and numbered for easy reference.)
$x$ lacks unit 0.1
Other than 0 or 1 decimal in $x \quad 0.1$
$x$ does not cover the interval $10 \mathrm{~mm}-160 \mathrm{~mm} \quad 0.1$
$T$ lacks unit 0.1
$T$ given with other than 1 or 0.5 millisecond accuracy 0.1
Fewer than 11 measuring points ( 15 mm sep.). Subtr. up to 0.2
Systematic error in $x$ (e.g. if measured from the top of the nut so that the

$$
\text { first } x=0 \mathrm{~mm}) \quad 0.2
$$

If not aware of doubling of the timer period 0.2
Other (specify):

$$
\text { Score for subsection 1a: } 1.0 \text { - = }
$$

| Subsection 1b) | Deficiency | Subtract |
| ---: | ---: | ---: |
| No answer | $\mathbf{1 . 0}$ |  |
| Lacks " $x[(\mathrm{~m}) \mathrm{m}]$ " on horizontal axis | 0.1 |  |
| 1 mm on paper does not correspond to 1 mm in $x$ | 0.1 |  |
| Fewer than 3 numbers on horizontal axis | 0.1 |  |
| Lacks " $T[(\mathrm{~m}) \mathrm{s}]$ " on vertical axis | 0.1 |  |
| 1 mm on paper does not correspond to 1 ms in $T$ | 0.1 |  |
| Fewer than 3 numbers on vertical axis | 0.1 |  |
| Measuring points not clearly shown (as circles or crosses) | 0.2 |  |
| More than 5 ms deviation in more than 2 measuring points on the graph | 0.2 |  |
| Wrong answer to the questions ( $x$-values give full score if correct number |  |  |
| Other (specify): |  |  |

Score for subsection 1b): 1.0 -

Subsection 1c)

| Deficiency | Subtract |
| :--- | :--- |
| No answer | $\mathbf{2 . 0}$ |

$x$ outside the interval $81-87 \mathrm{~mm}$. Subtract up to 0.4 $x$ lacks unit 0.1
$x$ given more (or less) accurately than in whole millimeters 0.3
$l$ lacks unit 0.1
$l$ given more (or less) accurately than the nearest mm 0.3
Wrong formula (e.g. $l=200.0 \mathrm{~mm}-x$ ) or something other than $l=x_{\mathrm{CM}}-x \quad 0.6$
If it is not possible to see which method was used to find the center of mass 0.2 Other (specify):
Other than 4 regions are drawn ..... 0.5
Inaccurate drawing ( $> \pm 2 \mathrm{~mm}$ ) ..... 0.3Lacks the values $l_{1}=60 \mathrm{~mm}, l_{2}=35 \mathrm{~mm}$ on figure or text 0.3

Other (specify):
Score for subsection 2a): 1.5 -

Subsection 2b)
No answer 2.5
Lacks (derivation of) formula for $g \quad 0.3$
For INVERTED PENDULUM: Lacks figure 0.2
Values from possible new measurements not given 0.3
Incomplete calculations 0.3
If hard to see which method was used 0.4
Used the formula for INVERTED PENDULUM but read $l_{1}$ and $l_{2}$ from
graph in 1b) by a horizontal line for a certain $T \quad 1.5$
Used one of the other incorrect methods 2.0
Other than 3 (or 4) significant figures in the answer 0.3
$g$ lacks unit m/s ${ }^{2} \quad 0.1$
Other (specify):

$$
\text { Score for subsection 2b): } 2.5 \text { - }
$$

| Subsection 2c) | Deficiency | Subtract |
| ---: | ---: | ---: |
| No answer | 2.5 |  |
| Wrong expression for $\Delta g / g$ or $\Delta g$. Subtract up to | 0.5 |  |
| For INVERTED PENDULUM: If $0.3 \mathrm{~mm}>\Delta\left(l_{1}+l_{2}\right)>0.5 \mathrm{~mm}$ | 0.2 |  |
| For ALTERNATIVE 2c $-1:$ If $\Delta[]>0.1 \mathrm{~mm}$ | 0.2 |  |
| For INCORRECT 2c-1 and 2c-2: If $1 \mathrm{~mm}>\Delta l>2 \mathrm{~mm}$ | 0.2 |  |
| For INCORRECT 2c-3: If $0.3 \mathrm{~mm}>\Delta L>0.4 \mathrm{~mm}$ | 0.2 |  |
| For all methods: If $\Delta T \neq 1$ (or 0.5$) \mathrm{ms}$ | 0.2 |  |
| Error in the calculation of $\Delta g$ | 0.2 |  |
| Lacks answer including $g \pm \Delta g$ with 2 decimals | 0.2 |  |
| $g \pm \Delta g$ lacks unit | 0.1 |  |

Other (specify):

| Subsection 3a) | a) Deficiency | Subtract |
| :---: | :---: | :---: |
|  | No answer | 1.0 |
|  | Lacks point 3. cylindrical mirror | 0.3 |
|  | Lacks point 4. concave mirror | 0.3 |
|  | Includes other points (1, 2 or 5), subtract per wrong point: | 0.3 |
|  | Lacks value for radius of curvature | 0.4 |
|  | If $r<130 \mathrm{~mm}$ or $r>160 \mathrm{~mm}$, subtract up to | 0.2 |
|  | If $r$ is given more accurately than hole millimeters | 0.2 |
| Other (specify): |  |  |
| Score for subsection 3a): 1.0 - |  | = |
| Subsection 3b) | b) Deficiency | Subtract |
|  | No answer | 2.0 |
|  | Lacks side view figure | 0.6 |
| Errors or deficiencies in the side view figure. Subtract up to |  | 0.4 |
|  | Lacks top view figure | 0.6 |
| Errors or deficiencies in the top view figure. Subtract up to |  | 0.4 |
| Drawing shows light focused to a point |  | 0.3 |
| Drawing shows light spread out over an ill defined or wrongly shaped |  |  |
|  | surface | 0.3 |
|  | Line/surface is not horizontal | 0.2 |
|  | Line/point/surface not centered symmetrically on detector | 0.2 |
| Line/point/su | t/surface has length different from twice the width of the nut (i.e. outside the interval $10-30 \mathrm{~mm}$ ) | 0.1 |
| Other (specify): |  |  |
|  | Score for subsection 3b): 2.0 - | $=$ |

Other (specify):
Score for subsection 4a): 1.0 - =

| Subsection 4b) | Deficiency | Subtract |
| ---: | ---: | :--- |
| No answer | $\mathbf{1 . 5}$ |  |
| Forgotten $V_{o}$ or other errors in formula for $B$ | 0.2 |  |
| Lacks " $y[(\mathrm{~m}) \mathrm{m}]$ " on horizontal axis | 0.1 |  |
| Fewer than 3 numbers on horizontal axis | 0.1 |  |
| Lacks "B $[(\mathrm{m}) \mathrm{T}]$ " on vertical axis | 0.1 |  |
| Fewer than 3 numbers on vertical axis | 0.1 |  |
| Fewer than 9 measuring points. Subtract up to | 0.2 |  |
| Measuring points do not cover the interval $3.5 \mathrm{~mm}-26 \mathrm{~mm}$ | 0.2 |  |
| Measuring points not clearly shown (as circles or crosses) | 0.1 |  |

Error in data or unreasonably large spread in measuring points. Subtract
up to 0.5
Other (specify):

$$
\text { Score for subsection 4b): } 1.5 \text { - = }
$$

| Subsection 4c) | Deficiency | Subtract |
| ---: | ---: | :--- |
| No answer | $\mathbf{1 . 5}$ |  |
| Incorrect formula for $B_{o}$ | 0.3 |  |
| If used only one measuring point | 0.4 |  |
| If used untypical points on the graph | 0.3 |  |
| Errors in calculation of mean value for $B_{o}$ | 0.2 |  |
| $B_{o}$ lacks unit $T$ | 0.1 |  |
| Other than two significant figures in (the mean value of) $B_{o}$ | 0.2 |  |
| $B_{o}<0.4$ T or $B_{o}>0.7 \mathrm{~T}$. Subtract up to | 0.2 |  |

Other (specify):

Lacks " $\mathrm{z}[(\mathrm{m}) \mathrm{m}]$ " on horizontal axis 0.1
Fewer than 3 numbers on horizontal axis 0.1
Lacks " $T[(\mathrm{~m}) \mathrm{s}]$ " on vertical axis 0.1
Fewer than 3 numbers on vertical axis 0.1
Fewer than 8 measuring points. Subtract up to 0.2
Measuring points not clearly shown (as circles or crosses) 0.1
Measuring points do not cover the interval $5.5 \mathrm{~mm}-25 \mathrm{~mm} \quad 0.2$
Error in data (e.g. plotted $2 T$ instead of T ) or unreasonably large spread
in measuring points. Subtr. up to 0.5
Other (specify):

$$
\text { Score for subsection 5a): } \mathbf{1 . 0}-\quad=
$$

Subsection 5b)
Deficiency Subtract
No answer 3.0
Forgotten center of mass displacement in $l$ (used $l=100 \mathrm{~mm}$ ) $\quad 0.3$
Used ALTERNATIVE 5b-1 1.0
Lacks method for finding the proportionality factor $a \quad 2.5$
Not found correct proportionality factor $a \quad 0.3$
Used another point than the maximum of $f(z) \quad 0.1$
Incorrect reading of $f(z) \quad 0.1$
Used $M_{\text {ROD }}$ or another incorrect value for $M \quad 0.2$
Incorrect calculation of $\mu \quad 0.3$
$\mu$ lacks unit ( $\mathrm{Am}^{2}$ or $\mathrm{J} / \mathrm{T}$ ) $\quad 0.2$
More than 2 significant figures in $\mu \quad 0.3$
Other (specify):
Score for subsection 5b): 3.0- =

## Total points:

Total for section 1 (max. 4 points):
Total for section 2 (max. 5 points):
Total for section 3 (max. 3 points):
Total for section 4 (max. 4 points):
Total for section 5 (max. 4 points):

## The last preparations

The problem for the experimental competition was discussed by the leaders and the organizers the evening before the exam. At this meeting the equipment was demonstrated for the first time (picture).


After the meeting had agreed on the final text (in English), the problems had to be translated into the remaining 36 languages. One PC was available for each nation for the translation process (see picture below). The last nation finished their translation at about 4:30 a.m. in the morning, and the competition started at 0830. Busy time for the organizers! Examples of the different translations are given on the following pages.


Photo: Børge Holme

# $28^{\text {th }}$ International Physics Olympiad Sudbury, Canada 

## THEORETICAL COMPETITION

Thursday, July $17^{\text {th }}, 1997$

Time Available: 5 hours

## Read This First:

1. Use only the pen provided.
2. Use only the front side of the answer sheets and paper.
3. In your answers please use as little text as possible; express yourself primarily in equations, numbers and figures. Summarize your results on the answer sheet.
4. Please indicate on the first page the total number of pages you used.
5. At the end of the exam please put your answer sheets, pages and graphs in order.

## This set of problems consists of 11 pages.

Examination prepared at: University of British Columbia
Department of Physics and Astronomy
Committee Chair: Chris Waltham

Hosted by: Laurentian University

## Theory Question No. 1

## Scaling

(a) A small mass hangs on the end of a massless ideal spring and oscillates up and down at its natural frequency $f$. If the spring is cut in half and the mass reattached at the end, what is the new frequency $f^{\prime}$ ? ( 1.5 marks)
(b) The radius of a hydrogen atom in its ground state is $a_{0}=0.0529 \mathrm{~nm}$ (the "Bohr radius"). What is the radius $a^{\prime}$ of a "muonic-hydrogen" atom in which the electron is replaced by an identically charged muon, with mass 207 times that of the electron? Assume the proton mass is much larger than that of the muon and electron. (2 marks)
(c) The mean temperature of the earth is $T=287 \mathrm{~K}$. What would the new mean temperature $T^{\prime}$ be if the mean distance between the earth and the sun was reduced by $1 \%$ ?
(2 marks)
(d) On a given day, the air is dry and has a density $\rho=1.2500 \mathrm{~kg} / \mathrm{m}^{3}$. The next day the humidity has increased and the air is $2 \%$ by mass water vapour. The pressure and temperature are the same as the day before. What is the air density $\rho^{\prime}$ now? marks)

Mean molecular weight of dry air: $28.8(\mathrm{~g} / \mathrm{mol})$
Molecular weight of water: $18(\mathrm{~g} / \mathrm{mol})$
Assume ideal-gas behaviour.
(e) A type of helicopter can hover if the mechanical power output of its engine is $P$. If another helicopter is made which is an exact $1 / 2$-scale replica (in all linear dimensions) of the first, what mechanical power $P^{\prime}$ is required for it to hover? (2.5 marks)
$\qquad$
(a) Frequency $f^{\prime}$ :
(b) Radius $a^{\prime}$ :
(c) Temperature $T^{\prime}$ :
(d) Density $\rho^{\prime}$ :
(e) Power $P^{\prime}$ :

## Theory Question No. 2

## Nuclear Masses and Stability

All energies in this question are expressed in MeV - millions of electron volts.
One $\mathrm{MeV}=1.6 \times 10^{-13} \mathrm{~J}$, but it is not necessary to know this to solve the problem.
The mass $M$ of an atomic nucleus with $Z$ protons and $N$ neutrons (i.e. the mass number $A=N+Z$ ) is the sum of masses of the free constituent nucleons (protons and neutrons) minus the binding energy $B / c^{2}$.

$$
M c^{2}=Z m_{p} c^{2}+N m_{n} c^{2}-B
$$

The graph shown below plots the maximum value of $B / A$ for a given value of $A$, vs. $A$. The greater the value of $B / A$, in general, the more stable is the nucleus.

Binding Energy per Nucleon

(a) Above a certain mass number $A_{\alpha}$, nuclei have binding energies which are always small enough to allow the emission of alpha-particles $(A=4)$. Use a linear approximation to this curve above $A=100$ to estimate $A_{\alpha}$. (3 marks)

For this model, assume the following:

- Both initial and final nuclei are represented on this curve.
- The total binding energy of the alpha-particle is given by $\mathrm{B}_{4}=25.0 \mathrm{MeV}$ (this cannot be read off the graph!).
(b) The binding energy of an atomic nucleus with $Z$ protons and $N$ neutrons $(A=N+Z)$ is given by a semi-empirical formula:

$$
B=a_{v} A-a_{s} A^{2 / 3}-a_{c} Z^{2} A^{-1 / 3}-a_{a} \frac{(N-Z)^{2}}{A}-\delta
$$

The value of $\delta$ is given by:

$$
+a_{p} A^{-3 / 4} \text { for odd-N/odd-Z nuclei }
$$

0 for even-N/odd-Z or odd-N/even-Z nuclei

$$
-a_{p} A^{-3 / 4} \text { for even-N/even-Z nuclei }
$$

The values of the coefficients are:
$a_{v}=15.8 \mathrm{MeV} ; a_{s}=16.8 \mathrm{MeV} ; a_{c}=0.72 \mathrm{MeV} ; a_{a}=23.5 \mathrm{MeV} ; a_{p}=33.5 \mathrm{MeV}$.
(i) Derive an expression for the proton number $Z_{\max }$ of the nucleus with the largest binding energy for a given mass number $A$. Ignore the $\delta$-term for this part only. (2 marks)
(ii) What is the value of $Z$ for the $A=200$ nucleus with the largest $B / A$ ? Include the effect of the $\delta$-term. ( 2 marks)
(iii) Consider the three nuclei with $\mathrm{A}=128$ listed in the table on the answer sheet. Determine which ones are energetically stable and which ones have sufficient energy to decay by the processes listed below. Determine $Z_{\max }$ as defined in part (i) and fill out the table on your answer sheet.

In filling out the table, please:

- Mark processes which are energetically allowed thus: $\sqrt{ }$
- Mark processes which are NOT energetically allowed thus: 0
- Consider only transitions between these three nuclei.

Decay processes:
(1) $\beta^{-}$- decay; emission from the nucleus of an electron
(2) $\beta^{+}$- decay; emission from the nucleus of a positron
(3) $\beta^{-} \beta^{-}$- decay; emission from the nucleus of two electrons simultaneously
(4) Electron capture; capture of an atomic electron by the nucleus.

The rest mass energy of an electron (and positron) is $m_{e} c^{2}=0.51 \mathrm{MeV}$; that of a proton is $m_{p} c^{2}=938.27 \mathrm{MeV}$; that of a neutron is $m_{n} c^{2}=939.57 \mathrm{MeV}$.
(3 marks)

Question 2: Answer Sheet $\qquad$
(a) Numerical value for $A_{\alpha}$ :
(b) (i) Expression for $Z_{\max }$ :
(b) (ii) Numerical value of $Z$ :
(b) (iii)

| Nucleus/Process | $\beta^{-}$- decay | $\beta^{+}$- decay | Electron-capture | $\beta^{-} \beta^{-}$- decay |
| :---: | :---: | :---: | :---: | :---: |
| ${ }_{53}^{128} \mathrm{I}$ |  |  |  |  |
| ${ }_{54}^{128} \mathrm{Xe}$ |  |  |  |  |
| ${ }_{55}^{18} \mathrm{Cs}$ |  |  |  |  |

Notation: $\quad{ }_{Z}^{A} X$

$$
\mathrm{X}=\text { Chemical Symbol }
$$

## Theory Question No. 3

## Solar-Powered Aircraft

We wish to design an aircraft which will stay aloft using solar power alone. The most efficient type of layout is one with a wing whose top surface is completely covered in solar cells. The cells supply electrical power with which the motor drives the propeller.

Consider a wing of rectangular plan-form with span $l$, chord (width) $c$; the wing area is $S=c l$, and the wing aspect ratio $A=l / c$. We can get an approximate idea of the wing's performance by considering a slice of air of height $x$ and length $l$ being deflected downward at a small angle $\varepsilon$ with only a very small change in speed. Control surfaces can be used to select an optimal value of $\varepsilon$ for flight. This simple model corresponds closely to reality if $x=\pi l / 4$, and we can assume this to be the case. The total mass of the aircraft is $M$ and it flies horizontally with velocity $\vec{v}$ relative to the surrounding air. In the following calculations consider only the air flow around the wing.

Top view of aircraft (in its own frame of reference):


Side view of wing (in a frame of reference moving with the aircraft):


$$
\text { incident air } \quad \text { wing section } \quad \text { air leaving wing vertical(up) }
$$

Ignore the modification of the airflow due to the propeller.
(a) Consider the change in momentum of the air moving past the wing, with no change in speed while it does so. Derive expressions for the vertical lift force $L$ and the horizontal drag force $D_{l}$ on the wing in terms of wing dimensions, $v, \varepsilon$, and the air density $\rho$. Assume the direction of air flow is always parallel to the plane of the side-view diagram. (3 marks)
(b) There is an additional horizontal drag force $D_{2}$ caused by the friction of air flowing over the surface of the wing. The air slows slightly, with a change of speed $\Delta v(\ll 1 \%$ of $v)$ given by:

$$
\frac{\Delta v}{v}=\frac{f}{A}
$$

The value of $f$ is independent of $\varepsilon$.
Find an expression (in terms of $M, f, A, S, \rho$ and $g$ - the acceleration due to gravity) for the flight speed $v_{0}$ corresponding to a minimum power being needed to maintain this aircraft in flight at constant altitude and velocity. Neglect terms of order $\left(\varepsilon^{2} f\right)$ or higher.
(3 marks)
You may find the following small angle approximation useful:

$$
1-\cos \varepsilon \approx \frac{\sin ^{2} \varepsilon}{2}
$$

(c) On the answer sheet, sketch a graph of power $P$ versus flight speed $v$. Show the separate contributions to the power needed from the two sources of drag. Find an expression (in terms of $M, f, A, S, \rho$ and $g$ ) for the minimum power, $P_{\text {min }}$.
(d) If the solar cells can supply sufficient energy so that the electric motors and propellers generate mechanical power of $I=10$ watts per square metre of wing area, calculate the maximum wing loading $M g / S\left(\mathrm{~N} / \mathrm{m}^{2}\right)$ for this power and flight speed $v_{0}(\mathrm{~m} / \mathrm{s})$. Assume $\rho=1.25 \mathrm{~kg} / \mathrm{m}^{3}, f=0.004, A=10$.
(2 marks)
$\qquad$
(a) Expression for $L$ :
(a) Expression for $D_{l}$ :
(b) Expression for $D_{2}$ :
(b) Expression for $v_{0}$ :
(c)

(c) Expression for $P_{\text {min }}$ :
(d) Maximum value of $M g / S$ :
(d) Numerical value of $v_{0}$ :

# $29^{\text {th }}$ International Physics Olympiad Reykjavík, Iceland 

Theoretical competition

Saturday, July 4 ${ }^{\text {th }}$, 1998

9 a.m. - 2 p.m.

## Read this first:

1. Use only the pen provided.
2. Use only the front side of the answer sheets.
3. Use as little text as possible in your answers; express yourself primarily with equations, numbers and figures. Summarize your results on the answer sheets.
4. For anything but your answers and your graphs use the blank answer sheets. This applies e.g. when you are asked to show that ... and also for all calculations you want to be considered for evaluation.
5. You may often be able to solve later parts of a problem without having solved the previous ones. In such cases you may take the result of a previous part as given, in the form stated in the problem text.
6. Please indicate on all sheets your team name, student number, number of page and total number of pages. On the blank answer sheets also indicate the problem number.
7. At the end of the exam please put your answer sheets in order. You may leave on your table material which you do not wish to be evaluated.

## This set of problems consists of 11 pages in total.

Examination prepared at:
University of Iceland, Department of Physics, in collaboration with physicists from the National Energy Authority.

### 1.1 Problem text

Consider a long, solid, rigid, regular hexagonal prism like a common type of pencil (Figure 1.1). The mass of the prism is $M$ and it is uniformly distributed. The length of each side of the cross-sectional hexagon is $a$. The moment of inertia $I$ of the hexagonal prism about its central axis is

$$
\begin{equation*}
I=\frac{5}{12} M a^{2} \tag{1.1}
\end{equation*}
$$



Figure 1.1: A solid prism with the cross section of a regular hexagon.
The moment of inertia $I^{\prime}$ about an edge of the prism is

$$
\begin{equation*}
I^{\prime}=\frac{17}{12} M a^{2} \tag{1.2}
\end{equation*}
$$

a) (3.5 points) The prism is initially at rest with its axis horizontal on an inclined plane which makes a small angle $\theta$ with the horizontal (Figure 1.2). Assume that the surfaces of the prism are slightly concave so that the prism only touches the plane at its edges. The effect of this concavity on the moment of inertia can be ignored. The prism is now displaced from rest and starts an uneven rolling down the plane. Assume that friction prevents any sliding and that the prism does not lose contact with the plane. The angular velocity just before a given edge hits the plane is $\omega_{i}$ while $\omega_{f}$ is the angular velocity immediately after the impact.

Show that we may write

$$
\begin{equation*}
\omega_{f}=s \omega_{i} \tag{1.3}
\end{equation*}
$$

and write the value of the coefficient $s$ on the answer sheet.

[^1]

Figure 1.2: A hexagonal prism lying on an inclined plane.
b) (1 point) The kinetic energy of the prism just before and after impact is similarly $K_{i}$ and $K_{f}$.

Show that we may write

$$
\begin{equation*}
K_{f}=r K_{i} \tag{1.4}
\end{equation*}
$$

and write the value of the coefficient $r$ on the answer sheet.
c) (1.5 points) For the next impact to occur $K_{i}$ must exceed a minimum value $K_{i, \text { min }}$ which may be written in the form

$$
\begin{equation*}
K_{i, \min }=\delta M g a \tag{1.5}
\end{equation*}
$$

where $g=9.81 \mathrm{~m} / \mathrm{s}^{2}$ is the acceleration of gravity.
Find the coefficient $\delta$ in terms of the slope angle $\theta$ and the coefficient $r$. Write your answer on the answer sheet. (Use the algebraic symbol $r$, not its value).
d) (2 points) If the condition of part (c) is satisfied, the kinetic energy $K_{i}$ will approach a fixed value $K_{i, 0}$ as the prism rolls down the incline.

Given that the limit exists, show that $K_{i, 0}$ may be written as:

$$
\begin{equation*}
K_{i, 0}=\kappa M g a \tag{1.6}
\end{equation*}
$$

and write the coefficient $\kappa$ in terms of $\theta$ and $r$ on the answer sheet.
e) (2 points) Calculate, to within $0.1^{\circ}$, the minimum slope angle $\theta_{0}$, for which the uneven rolling, once started, will continue indefinitely. Write your numerical answer on the answer sheet.

### 1.2 Solution

a)

Solution Method 1

At the impact the prism starts rotating about a new axis, i.e. the edge which just hit the plane. The force from the plane has no torque about this axis, so that the angular momentum about the edge is conserved during the brief interval of impact. The linear
momentum of the prism as a whole has the same direction as the velocity of the center of mass ( $\vec{P}=M \vec{v}_{C}$ where the subscript $C$ refers to the center of mass), and this direction is easy to follow when we know the axis of rotation at a given time. Just before impact $\vec{P}$ is directed $30^{\circ}$ downwards relative to the plane, but will after impact point $30^{\circ}$ upwards from the plane, see Figure 1.3.


Figure 1.3: The linear momentum of the prism as a whole, before and after impact.
To find the angular momentum about the edge of impact just before the impact we use the equation relating angular momentum $\vec{L}$ about an arbitrary axis to the angular momentum $\vec{L}_{C}$ about an axis through the center of mass parallel to the first one:

$$
\begin{equation*}
\vec{L}=\vec{L}_{C}+M \vec{r}_{C} \times \vec{v}_{C} \tag{1.7}
\end{equation*}
$$

where the subscript $C$ refers to the center of mass. Here, this is applied to an axis at the point of impact so that $\vec{r}_{C}$ is the vector from that point to the center of mass (Figure 1.3). The vectors on the right hand side of equation (1.7) both have the same direction. Hence we get for the quantities just before impact ${ }^{2}$

$$
\begin{align*}
\left|\vec{r}_{C} \times \vec{v}_{C i}\right| & =r_{C} v_{C i} \sin 30^{\circ}=a^{2} \omega_{i} / 2  \tag{1.8}\\
L_{i}=I \omega_{i}+\frac{1}{2} M a^{2} \omega_{i} & =\left(\frac{5}{12}+\frac{1}{2}\right) M a^{2} \omega_{i}=\frac{11}{12} M a^{2} \omega_{i} \tag{1.9}
\end{align*}
$$

On the other hand, angular momentum about the edge just after impact is, from equation (1.2): ${ }^{3}$

[^2]\[

$$
\begin{equation*}
L_{f}=I^{\prime} \omega_{f}=\frac{17}{12} M a^{2} \omega_{f} \tag{1.10}
\end{equation*}
$$

\]

where the subscript $f$ always refers to the situation just after impact. We may notice that the difference comes about because of the different directions of $\vec{v}_{C i}$ and $\vec{v}_{C f}$. Now, when we state the conservation of angular momentum, $L_{i}=L_{f}$, we obtain a relation between the angular velocities as follows:

$$
\begin{equation*}
\omega_{f}=\frac{11 / 12}{17 / 12} \omega_{i}=\frac{11}{17} \omega_{i} \tag{1.11}
\end{equation*}
$$

We thus get:

$$
\begin{equation*}
s=11 / 17 \tag{1.12}
\end{equation*}
$$

We may note that $s$ is independent of $a, \omega_{i}$, and $\theta$.

## Solution Method 2

On impact the prism receives an impulse $\vec{P}[\mathrm{~N} \cdot \mathrm{~s}]$ from the plane at the edge where the impact occurs. There is no reaction at the edge which is leaving the plane. The impulse has a component $P_{\|}$parallel to the inclined plane (positive upwards along the incline in Figure 1.3 and a component $P_{\perp}$ perpendicular to the plane (positive upwards from the plane in the same figure).

We can set up three equations with the three unknowns $P_{\|}, P_{\perp}$ and the ratio $s=\frac{\omega_{f}}{\omega_{i}}$. The quantity $P_{\| \mid}$is the change in the parallel component of the linear momentum of the prism and $P_{\perp}$ is the corresponding change in perpendicular linear momentum. Thus:

$$
\begin{align*}
P_{\|} & =M\left(\omega_{i}-\omega_{f}\right) a \cdot \frac{\sqrt{3}}{2}  \tag{1.13}\\
P_{\perp} & =M\left(\omega_{i}+\omega_{f}\right) a \cdot \frac{1}{2} \tag{1.14}
\end{align*}
$$

We finally have:

$$
\begin{equation*}
P_{\perp} a \frac{1}{2}-P_{\|} a \frac{\sqrt{3}}{2}=I\left(\omega_{i}-\omega_{f}\right) \tag{1.15}
\end{equation*}
$$

since the right hand side is the change in angular momentum about the center of mass. Equations (1.13), (1.14) and (1.15) can now be solved for the ratio $s=\frac{\omega_{f}}{\omega_{i}}$ giving, of course, the same result as before.

$$
\begin{aligned}
L_{f} & =I \omega_{f}+M\left|\vec{r}_{C} \times \vec{v}_{C f}\right|=I \omega_{f}+M a^{2} \omega_{f} \sin 90^{\circ} \\
& =\left(\frac{5}{12}+1\right) M a^{2} \omega_{f}=\frac{17}{12} M a^{2} \omega_{f}
\end{aligned}
$$

The linear speed of the center of mass just before impact is $a \omega_{i}$ and just after impact it is $a \omega_{f}$. We know that we can always write the kinetic energy of a rotating rigid body as a sum of ,internal" and „external" kinetic energy:

$$
\begin{equation*}
K_{t o t}=\frac{1}{2} I \omega^{2}+\frac{1}{2} M v_{C}^{2} \tag{1.16}
\end{equation*}
$$

From this we see that in our case the kinetic energy $K_{\text {tot }}$ is proportional to $\omega^{2}$ both before and after impact so that we get

$$
\begin{equation*}
K_{f}=r K_{i}=\left(\frac{11}{17}\right)^{2} K_{i}=\frac{121}{289} K_{i} \tag{1.17}
\end{equation*}
$$

so

$$
\begin{equation*}
r=121 / 289 \approx 0.419 \tag{1.18}
\end{equation*}
$$

c)

The kinetic energy $K_{f}$ after the impact must be sufficient to lift the center of mass to its highest position, straight above the point of contact. The angle through which $\vec{r}_{C}$ moves for this is

$$
\begin{equation*}
x=\frac{\alpha}{2}-\theta \tag{1.19}
\end{equation*}
$$

where $\alpha=60^{\circ}$ is the top angle of the triangles meeting at the center of the polygon. ${ }^{4}$ The energy for this lifting of the center of mass is

$$
\begin{equation*}
E_{0}=M g a(1-\cos x)=M g a\left(1-\cos \left(30^{\circ}-\theta\right)\right) \tag{1.20}
\end{equation*}
$$

and we get the condition

$$
\begin{equation*}
K_{f}=r K_{i}>E_{0}=M g a\left(1-\cos \left(30^{\circ}-\theta\right)\right) \tag{1.21}
\end{equation*}
$$

thus

$$
\begin{equation*}
\delta=\frac{1}{r}\left(1-\cos \left(30^{\circ}-\theta\right)\right) \tag{1.22}
\end{equation*}
$$

(Note that $\left.\cos \left(30^{\circ}-\theta\right)=\frac{\sqrt{3}}{2} \cos \theta+\frac{1}{2} \sin \theta\right)$.
d)

Let $K_{i, n}$ and $K_{f, n}$ be the kinetic energies just before and just after the $n$th impact. We have shown that we have the relation

[^3]\[

$$
\begin{equation*}
K_{f, n}=r K_{i, n} \tag{1.23}
\end{equation*}
$$

\]

where $r=\frac{121}{289}$ for a hexagonal prism. Between subsequent impacts the height of the center of mass of the prism decreases by $a \sin \theta$ and its kinetic energy increases for this reason by

$$
\begin{equation*}
\Delta=M g a \sin \theta \tag{1.24}
\end{equation*}
$$

We therefore have

$$
\begin{equation*}
K_{i, n+1}=r K_{i, n}+\Delta \tag{1.25}
\end{equation*}
$$

One does not have to write out the complete expression $K_{i, n}$ as a function of $K_{i, 1}$ and $n$ to find the limit. This would actually be a proof that the limit exists (see below) but this is given in the problem text. Hence one can make $K_{i, n+1} \approx K_{i, n}$ arbitrarily accurate for sufficiently large $n$. The limit $K_{i, 0}$ must thus satisfy the iterative formula, i.e.

$$
\begin{equation*}
K_{i, 0}=r K_{i, 0}+\Delta \tag{1.26}
\end{equation*}
$$

yielding the solution

$$
\begin{equation*}
K_{i, 0}=\frac{\Delta}{1-r} . \tag{1.27}
\end{equation*}
$$

i.e.

$$
\begin{equation*}
\kappa=\frac{\sin \theta}{1-r} \tag{1.28}
\end{equation*}
$$

We can also solve the problem explicitly by writing out the full expressions:

$$
\begin{align*}
K_{i, 2} & =r K_{i, 1}+\Delta  \tag{1.29}\\
K_{i, 3} & =r K_{i, 2}+\Delta=r^{2} K_{i, 1}+(1+r) \Delta  \tag{1.30}\\
\ldots &  \tag{1.31}\\
K_{i, n} & =r^{n-1} K_{i, 1}+\left(1+r+\ldots+r^{n-2}\right) \Delta  \tag{1.32}\\
& =r^{n-1} K_{i, 1}+\frac{1-r^{n-1}}{1-r} \Delta
\end{align*}
$$

In the limit of $n \rightarrow \infty$ we get

$$
\begin{equation*}
K_{i, n} \rightarrow K_{i, 0}=\frac{\Delta}{1-r} \tag{1.33}
\end{equation*}
$$

which is, of course, the same result as before.
If we calculate the change in kinetic energy through a whole cycle, i.e. from just before impact number $n$ until just before impact $n+1$ we get

$$
\begin{align*}
\Delta K_{i, n}=K_{i, n+1}-K_{i, n} & =(r-1) r^{n-1} K_{i, 1}+r^{n-1} \Delta  \tag{1.34}\\
& =r^{n-1}\left(\Delta-(1-r) K_{i, 1}\right) \tag{1.35}
\end{align*}
$$

This is positive if the initial value $K_{i, 1}<K_{i, 0}$ so that $K_{i, n}$ will then increase up to the limit value $K_{i, 0}$. If, on the other hand, $K_{i, 1}>K_{i, 0}$, the kinetic energy $K_{i, n}$ just before impact will decrease down to the limit $K_{i, 0}$.

All of this may remind you of motion with friction which increases with speed. Mathematically speaking, the main difference is that we here are dealing with difference equations instead of differential equations.
e)

For indefinite continuation the limit value of $K_{i}$ in part (d) must be larger than the minimum value for continuation found in part (c):

$$
\begin{equation*}
\frac{1}{1-r} \Delta=\frac{1}{1-r} M g a \sin \theta>M g a\left(1-\cos \left(30^{\circ}-\theta\right)\right) / r \tag{1.36}
\end{equation*}
$$

We put $A=\frac{r}{1-r}=\frac{121}{168}$ :

$$
\begin{array}{r}
A \sin \theta>1-\cos 30^{\circ} \cos \theta-\sin 30^{\circ} \sin \theta \\
(A+1 / 2) \sin \theta+\sqrt{3} / 2 \cos \theta>1 \tag{1.38}
\end{array}
$$

To solve this we define ${ }^{5}$

$$
\begin{equation*}
u=\arccos \left(\frac{A+1 / 2}{\sqrt{(A+1 / 2)^{2}+3 / 4}}\right) \approx 35.36^{\circ} \tag{1.39}
\end{equation*}
$$

and obtain

$$
\begin{align*}
\cos u \sin \theta+\sin u \cos \theta & >1 / \sqrt{(A+1 / 2)^{2}+3 / 4}  \tag{1.40}\\
\sin (u+\theta) & >1 / \sqrt{(A+1 / 2)^{2}+3 / 4}  \tag{1.41}\\
\theta>\arcsin \left\{1 / \sqrt{(A+1 / 2)^{2}+3 / 4}\right\}-u & \approx 41.94^{\circ}-35.36^{\circ}=6.58^{\circ} \tag{1.42}
\end{align*}
$$

That is

$$
\begin{equation*}
\theta_{0} \approx 6.58^{\circ} \tag{1.43}
\end{equation*}
$$

If $\theta>\theta_{0}$ and the kinetic energy before the first impact is sufficient according to part (c), we will, under the assumptions made, get an indefinite "rolling".

[^4]1.3 Grading scheme

| Part 2(a) |  |
| :--- | :---: |
| Answer: $s=\omega_{f} / \omega_{i}=11 / 17$, equation (1.12) | $\mathbf{3 . 5}$ |
| Part 2(b) |  |
| Answer: $r=K_{f} / K_{i}=s^{2}=121 / 289$, equation (1.18) | $\mathbf{1 . 0}$ |
| Part 2(c) | $\mathbf{1 . 5}$ |
| Answer: $K_{i, \min }$ by $\delta$, equation (1.22) | $\mathbf{2 . 0}$ |
| Answer: Limit $K_{i, 0}$ by $\kappa=\sin \theta /(1-r)$, equation $(1.28)$ | $\mathbf{2 . 0}$ |
| Part 2(e) |  |
| Answer: Minimum angle $\theta_{0}=6.58^{\circ}$, equation $(1.43)$ |  |

## 2 Water under an ice cap ${ }^{6}$

### 2.1 Problem text

An ice cap is a thick sheet of ice (up to a few km in thickness) resting on the ground below and extending horizontally over tens or hundreds of km . In this problem we consider the melting of ice and the behavior of water under a temperate ice cap, i.e. an ice cap at the melting point. We may assume that under such conditions the ice causes pressure variations as a viscous fluid, but deforms in a brittle fashion, principally by vertical movement. For the purposes of this problem the following information is given.

|  |  |
| :--- | :--- |
| Density of water: | $\rho_{w}=1.000 \cdot 10^{3} \mathrm{~kg} / \mathrm{m}^{3}$ |
| Density of ice: | $\rho_{i}=0.917 \cdot 10^{3} \mathrm{~kg} / \mathrm{m}^{3}$ |
| Specific heat of ice: | $c_{i}=2.1 \cdot 10^{3} \mathrm{~J} /\left(\mathrm{kg}{ }^{\circ} \mathrm{C}\right)$ |
| Specific latent heat of ice: | $L_{i}=3.4 \cdot 10^{5} \mathrm{~J} / \mathrm{kg}$ |
| Density of rock and magma: | $\rho_{r}=2.9 \cdot 10^{3} \mathrm{~kg} / \mathrm{m}^{3}$ |
| Specific heat of rock and magma: | $c_{r}=700 \mathrm{~J} /\left(\mathrm{kg}{ }^{\circ} \mathrm{C}\right.$ |
| Specific latent heat of rock and magma: | $L_{r}=4.2 \cdot 10^{5} \mathrm{~J} / \mathrm{kg}$ |
| Average outward heat flow through the | $J_{Q}=0.06 \mathrm{~W} / \mathrm{m}^{2}$ |
| surface of the earth: | $T_{0}=0^{\circ} \mathrm{C}$, constant |
| Melting point of ice: |  |

a) (0.5 points) Consider a thick ice cap at a location of average heat flow from the interior of the earth. Using the data from the table, calculate the thickness $d$ of the ice layer melted every year and write your answer in the designated box on the answer sheet.
b) (3.5 points) Consider now the upper surface of an ice cap. The ground below the ice cap has a slope angle $\alpha$. The upper surface of the cap slopes by an angle $\beta$ as shown in Figure 2.1. The vertical thickness of the ice at $x=0$ is $h_{0}$. Hence the lower and upper surfaces of the ice cap can be described by the equations

$$
\begin{equation*}
y_{1}=x \tan \alpha, y_{2}=h_{0}+x \tan \beta \tag{2.1}
\end{equation*}
$$

Derive an expression for the pressure $p$ at the bottom of the ice cap as a function of the horizontal coordinate $x$ and write it on the answer sheet.

Formulate mathematically a condition between $\beta$ and $\alpha$, so that water in a layer between the ice cap and the ground will flow in neither direction. Show that the condition is of the form $\tan \beta=s \tan \alpha$. Find the coefficient $s$ and write the result in a symbolic form on the answer sheet.

The line $y_{1}=0.8 x$ in Figure 2.2 shows the surface of the earth below an ice cap. The vertical thickness $h_{0}$ at $x=0$ is 2 km . Assume that water at the bottom is in equilibrium.

On a graph answer sheet draw the line $y_{1}$ and add a line $y_{2}$ showing the upper surface of the ice. Indicate on the figure which line is which.

[^5]

Figure 2.1: Cross section of an ice cap with a plane surface resting on an inclined plane ground. $S$ : surface, $G$ : ground, $I$ : ice cap.
c) (1 point) Within a large ice sheet on horizontal ground and originally of constant thickness $D=2.0 \mathrm{~km}$, a conical body of water of height $H=1.0 \mathrm{~km}$ and radius $r=1.0$ km is formed rather suddenly by melting of the ice (Figure 2.3). We assume that the remaining ice adapts to this by vertical motion only.

Show analytically on a blank answer sheet and pictorially on a graph answer sheet, the shape of the surface of the ice cap after the water cone has formed and hydrostatic equilibrium has been reached.
d) (5 points) In its annual expedition an international group of scientists explores a temperate ice cap in Antarctica. The area is normally a wide plateau but this time they find a deep crater-like depression, formed like a top-down cone with a depth $h$ of 100 m and a radius $r$ of 500 m (Figure 2.4). The thickness of the ice in the area is 2000 m .

After a discussion the scientists conclude that most probably there was a minor volcanic eruption below the ice cap. A small amount of magma (molten rock) intruded at the bottom of the ice cap, solidified and cooled, melting a certain volume of ice. The scientists try as follows to estimate the volume of the intrusion and get an idea of what became of the melt water.

Assume that the ice only moved vertically. Also assume that the magma was completely molten and at $1200^{\circ} \mathrm{C}$ at the start. For simplicity, assume further that the intrusion had the form of a cone with a circular base vertically below the conical depression in the surface. The time for the rising of the magma was short relative to the time for the exchange of heat in the process. The heat flow is assumed to have been primarily vertical such that the volume melted from the ice at any time is bounded by a conical surface centered above the center of the magma intrusion.

Given these assumptions the melting of the ice takes place in two steps. At first the water is not in pressure equilibrium at the surface of the magma and hence flows away. The water flowing away can be assumed to have a temperature of $0^{\circ} \mathrm{C}$. Subsequently,


Figure 2.2: Cross section of a temperate ice cap resting on an inclined ground with water at the bottom in equilibrium. $G$ : ground, $I$ : ice cap.
hydrostatic equilibrium is reached and the water accumulates above the intrusion instead of flowing away.

When thermal equilibrium has been reached, you are asked to determine the following quantities. Write the answers on the answer sheet.

1. The height $H$ of the top of the water cone formed under the ice cap, relative to the original bottom of the ice cap.
2. The height $h_{1}$ of the intrusion.
3. The total mass $m_{\text {tot }}$ of the water produced and the mass $m^{\prime}$ of water that flows away.

Plot on a graph answer sheet, to scale, the shapes of the rock intrusion and of the body of water remaining. Use the coordinate system suggested in Figure 2.4.

### 2.2 Solution

a)

Based on the conservation of energy we have

$$
\begin{equation*}
J_{Q} \cdot 1 \text { year }=L_{i} \rho_{i} d \tag{2.2}
\end{equation*}
$$



Figure 2.3: A vertical section through the mid-plane of a water cone inside an ice cap. $S$ : surface, $W$ : water, $G$ : ground, $I$ : ice cap.

$$
\begin{equation*}
\boldsymbol{d}=\frac{J_{Q} \cdot 1 \text { year }}{L_{i} \rho_{i}}=\frac{0.06 \mathrm{~J} \mathrm{~s}^{-1} \mathrm{~m}^{-2} 365.25 \cdot 24 \cdot 60 \cdot 60 \mathrm{~s}}{3.4 \cdot 10^{5} \mathrm{~J} / \mathrm{kg} 917 \mathrm{~kg} / \mathrm{m}^{3}}=\mathbf{6 . 1} \cdot \mathbf{1 0} 0^{-\mathbf{3}} \mathbf{m} \tag{2.3}
\end{equation*}
$$

b)

Let $p_{a}$ be the atmospheric pressure, taken to be constant. At a depth $z$ inside the ice cap the pressure is given by:

$$
\begin{equation*}
p=\rho_{i} g z+p_{a} \tag{2.4}
\end{equation*}
$$

Therefore, at the bottom of the ice cap, where $z=y_{2}-y_{1}$ :

$$
\begin{align*}
\boldsymbol{p} & =\rho_{i} g\left(y_{2}-y_{1}\right)+p_{a}  \tag{2.5}\\
& =\boldsymbol{\rho}_{\boldsymbol{i}} \boldsymbol{g} \boldsymbol{x}(\tan \boldsymbol{\beta}-\tan \boldsymbol{\alpha})+\boldsymbol{\rho}_{\boldsymbol{i}} \boldsymbol{g} \boldsymbol{h}_{\mathbf{0}}+\boldsymbol{p}_{\boldsymbol{a}} \tag{2.6}
\end{align*}
$$

For water not to move at the base of the ice cap the pressure must be hydrostatic (trivial, but can be seen from Bernoulli's equation), i.e.


Figure 2.4: A vertical and central cross section of a conical depression in a temperate ice cap. S: surface, $G$ : ground, I: ice cap, M: rock/magma intrusion, W: water. Note that the figure is NOT drawn to scale.

$$
\begin{align*}
p & =\text { constant }-\rho_{w} g y_{1}  \tag{2.7}\\
& =\text { constant }-\rho_{w} g x \tan \alpha \tag{2.8}
\end{align*}
$$

Therefore

$$
\begin{equation*}
\rho_{i} g x(\tan \beta-\tan \alpha)=-\rho_{w} g x \tan \alpha \tag{2.9}
\end{equation*}
$$

leading to

$$
\begin{array}{r}
\tan \boldsymbol{\beta}=-\frac{\rho_{w}-\rho_{i}}{\rho_{i}} \tan \alpha=-\frac{\boldsymbol{\Delta} \boldsymbol{\rho}}{\boldsymbol{\rho}_{\boldsymbol{i}}} \tan \alpha \approx-0.091 \tan \alpha \\
s=-\boldsymbol{\Delta} \boldsymbol{\rho} / \boldsymbol{\rho}_{\boldsymbol{i}}=-\mathbf{0 . 0 9 1} \tag{2.11}
\end{array}
$$

where the minus-sign is significant.
This can also be seen in various ways by looking at a mass element of water at the bottom of the ice and demanding equilibrium. - We now proceed with the solution.

With $\tan \alpha=0.8$, we get $\tan \beta=-0.073$ and

$$
\begin{equation*}
y_{2}=2 \mathrm{~km}-0.073 x \tag{2.13}
\end{equation*}
$$

The students are supposed to draw this line on a graph.
c)

Since the ice adapts by vertical motion only we see that the conical depression at the surface will have the same radius of 1.0 km as the intrusion. According to (b) it will have a depth of

$$
\begin{align*}
h & =|r \tan \beta|=\frac{\Delta \rho}{\rho_{i}} r \tan \alpha  \tag{2.14}\\
& =\frac{\Delta \rho}{\rho_{i}} H  \tag{2.15}\\
& =0.091 \cdot 1 \mathrm{~km}=91 \mathrm{~m} . \tag{2.16}
\end{align*}
$$

The students are supposed to show this result as a graph.
d)

The volume of a circular cone is $V=\frac{1}{3} \pi r^{2} h$. We assume that the height of the intrusion is $h_{1}$. We may say that it firstly melts an ice cone of its own volume $V_{1}=\frac{1}{3} \pi r^{2} h_{1}$. Pressure equilibrium has not yet been reached. Hence the water will flow away and the ice will keep contact with the face of the intrusion making the upper surface of the ice horizontal again. The intrusion then melts a volume equivalent to a cone of height $h_{2}=\frac{\Delta \rho}{\rho_{i}} h_{1}$ whereupon pressure equilibrium has been reached (following part (c)). During this second phase the melted water will also flow away. Assuming that the intrusion still has not cooled down to $0^{\circ} \mathrm{C}$ the intrusion will further melt a volume equivalent to a cone of height $h_{3}$, its water accumulating in place, forming a cone of height $h_{3}^{\prime}=\frac{\rho_{i}}{\rho_{w}} h_{3}$ relative to the top of the intrusion. The total height of the ice cone melted is

$$
\begin{equation*}
h_{t o t}=h_{1}+h_{2}+h_{3} \tag{2.17}
\end{equation*}
$$

The depth of the depression at the surface will be given by

$$
\begin{equation*}
h=\frac{\Delta \rho}{\rho_{i}}\left(h_{1}+h_{3}^{\prime}\right) \tag{2.18}
\end{equation*}
$$

which is most easily seen by considering pressure equilibrium in the final situation (again following part (c)). Thus, the requested height of the top of the water cone is

$$
\begin{equation*}
\boldsymbol{H}=h_{1}+h_{3}^{\prime}=\frac{\rho_{i}}{\Delta \rho} h=\mathbf{1 . 1} \times \mathbf{1 0}^{\mathbf{3}} \mathbf{m} \tag{2.19}
\end{equation*}
$$

The heat balance gives

$$
\begin{equation*}
\frac{1}{3} \pi r^{2}\left\{\rho_{r} h_{1}\left(L_{r}+c_{r} \Delta T\right)-\rho_{i} L_{i} h_{t o t}\right\}=0 \tag{2.20}
\end{equation*}
$$

where $\Delta T=1200^{\circ} \mathrm{C}$ is the change in temperature of the rock intrusion. Following equation (2.17) and using the facts that $h_{2}=\frac{\Delta \rho}{\rho_{i}} h_{1}$ and $h_{3}=\frac{\rho_{w}}{\rho_{i}} h_{3}^{\prime}$ we obtain

$$
\begin{equation*}
h_{\text {tot }}=h_{1}+\frac{\Delta \rho}{\rho_{i}} h_{1}+\frac{\rho_{w}}{\rho_{i}} h_{3}^{\prime}=\frac{\rho_{w}}{\rho_{i}}\left(h_{1}+h_{3}^{\prime}\right) \tag{2.21}
\end{equation*}
$$

Therefore (using equation (2.19))

$$
\begin{equation*}
h_{t o t}=\frac{\rho_{w}}{\rho_{i}}\left(h_{1}+h_{3}^{\prime}\right)=\frac{\rho_{w}}{\rho_{i}} H=\frac{\rho_{w}}{\Delta \rho} h=1.20 \cdot 10^{3} \mathrm{~m} \tag{2.22}
\end{equation*}
$$

This implies that the cone does not reach the surface of the ice cap. Inserting the result into the equation (2.20) we can solve for $h_{1}$ :

$$
\begin{align*}
\rho_{r} h_{1} & \left(L_{r}+c_{r} \Delta T\right)=\frac{\rho_{i} \rho_{w} L_{i} h}{\Delta \rho}  \tag{2.23}\\
\boldsymbol{h}_{\mathbf{1}} & =\frac{\rho_{i} \rho_{w} L_{i} h}{\Delta \rho \rho_{r}\left(L_{r}+c_{r} \Delta T\right)}  \tag{2.24}\\
& =\mathbf{1 0 3} \mathbf{~ m} \tag{2.25}
\end{align*}
$$

The total mass of water formed is of course equal to the mass of the ice melted and is

$$
\begin{equation*}
\boldsymbol{m}_{\boldsymbol{t o t}}=\rho_{i}(1 / 3) \pi r^{2} h_{t o t}=\mathbf{2 . 9} \cdot \mathbf{1 0}^{\mathbf{1 1}} \mathbf{~ k g} \tag{2.26}
\end{equation*}
$$

The mass of the water which flows away is

$$
\begin{equation*}
\boldsymbol{m}^{\prime}=\frac{h_{1}+h_{2}}{h_{t o t}} m_{t o t}=\frac{\rho_{w} h_{1}}{\rho_{i} h_{t o t}} m_{t o t}=\mathbf{2 . 7} \cdot \mathbf{1 0}^{\mathbf{1 0}} \mathbf{~ k g} \tag{2.27}
\end{equation*}
$$

The students are finally expected to plot the shapes of the rock intrusion and the water body.

### 2.3 Grading scheme

| 2(a) |  |
| :---: | :---: |
| Answer: equation (2.3), $d=6.1 \cdot 10^{-3} \mathrm{~m}$ | 0.5 |
| 2(b) |  |
| Answer i): equation (2.6): $p=\rho_{i} g x(\tan \beta-\tan \alpha)+\rho_{i} g h_{0}+p_{a}$ | 1.0 |
| Answer ii): equation (2.10): $s=-\frac{\rho_{w}-\rho_{i}}{\rho_{i}}=-\frac{\Delta \rho}{\rho_{i}}$ | 2.0 |
| Answer iii): Graph based on equation (2.13) | 0.5 |
| 2(c) |  |
| Answer: Depth, radius and graph, $r=1000 \mathrm{~m}, h=91 \mathrm{~m}$ | 1.0 |
| 2(d) |  |
| Answer i): Height of water cone as in (2.19): $H=1.1 \cdot 10^{3} \mathrm{~m}$ | 2.0 |
| Answer ii): Height of intrusion as in (2.25): $h_{1}=103 \mathrm{~m}$ | 1.0 |
| Answer iii): Total mass of melt water as in (2.26): $m_{\text {tot }}=2.9 \cdot 10^{11} \mathrm{~kg}$ | 0.5 |
| Answer iv): Mass of water flowing away as in (2.27): $m^{\prime}=2.7 \cdot 10^{10} \mathrm{~kg}$ | 1.0 |
| Answer v): Graph | 0.5 |

## 3 Faster than light?

### 3.1 Problem text

In this problem we analyze and interpret measurements made in 1994 on radio wave emission from a compound source within our galaxy.

The receiver was tuned to a broad band of radio waves of wavelengths of several centimeters. Figure 3.1 shows a series of images recorded at different times. The contours indicate constant radiation strength in much the same way as altitude contours on a geographical map. In the figure the two maxima are interpreted as showing two objects moving away from a common center shown by crosses in the images. (The center, which is assumed to be fixed in space, is also a strong radiation emitter but mainly at other wavelengths). The measurements conducted on the various dates were made at the same time of day.

The scale of the figure is given by a line segment showing one arc second (as). (1 as = $1 / 3600$ of a degree). The distance to the celestial body at the center of the figure, indicated by crosses, is estimated to be $R=12.5 \mathrm{kpc}$. A kiloparsec ( kpc ) equals $3.09 \cdot 10^{19} \mathrm{~m}$. The speed of light is $c=3.00 \cdot 10^{8} \mathrm{~m} / \mathrm{s}$. Error calculations are not required in the solution.
a) (2 points) We denote the angular positions of the two ejected radio emitters, relative to the common center, by $\theta_{1}(t)$ and $\theta_{2}(t)$, where the subscripts 1 and 2 refer to the left and right hand ones, respectively, and $t$ is the time of observation. The angular speeds, as seen from the Earth, are $\omega_{1}$ and $\omega_{2}$. The corresponding apparent transverse linear speeds of the two sources are denoted by $v_{1, \perp}^{\prime}$ and $v_{2, \perp}^{\prime}$.

Using Figure 3.1, make a graph to find the numerical values of $\omega_{1}$ and $\omega_{2}$ in milli-arcseconds per day (mas/d). Also determine the numerical values of $v_{1, \perp}^{\prime}$ and $v_{2, \perp}^{\prime}$, and write all answers on the answer sheet. (You may be puzzled by some of the results).
b) (3 points) In order to resolve the puzzle arising in part (a), consider a light-source moving with velocity $\vec{v}$ at an angle $\phi(0 \leq \phi \leq \pi)$ to the direction towards a distant observer $O$ (Figure 3.2). The speed may be written as $v=\beta c$, where $c$ is the speed of light. The distance to the source, as measured by the observer, is $R$. The angular speed of the source, as seen from the observer, is $\omega$, and the apparent linear speed perpendicular to the line of sight is $v_{\perp}^{\prime}$.

Find $\omega$ and $v_{\perp}^{\prime}$ in terms of $\beta, R$ and $\phi$ and write your answer on the answer sheet.
c) (1 point) We assume that the two ejected objects, described in the introduction and in part (a), are moving in opposite directions with equal speeds $v=\beta c$. Then the results of part (b) make it possible to calculate $\beta$ and $\phi$ from the angular speeds $\omega_{1}$ and $\omega_{2}$ and the distance $R$. Here $\phi$ is the angle defined in part (b), for the left hand object, corresponding to subscript 1 in part (a).

Derive formulas for $\beta$ and $\phi$ in terms of known quantities and determine their numerical values from the data in part (a). Write your answers in the designated fields on the answer sheet.
d) (2 points) In the one-body situation of part (b), find the condition for the apparent perpendicular speed $v_{\perp}^{\prime}$ to be larger than the speed of light $c$.

[^6]

Figure 3.1: Radio emission from a source in our galaxy.


Figure 3.2: The observer is at $O$ and the original position of the light source is at $A$. The velocity vector is $\vec{v}$.

Write the condition in the form $\beta>f(\phi)$ and provide an analytic expression for the function $f$ on the answer sheet.

Draw on the graph answer sheet the physically relevant region of the ( $\beta, \phi$ )-plane. Show by shading in which part of this region the condition $v_{\perp}^{\prime}>c$ holds.
e) (1 point) Still in the one-body situation of part (b), find an expression for the maximum value $\left(v_{\perp}^{\prime}\right)_{\max }$ of the apparent perpendicular speed $v_{\perp}^{\prime}$ for a given $\beta$ and write it in the designated field on the answer sheet. Note that this speed increases without limit when $\beta \rightarrow 1$.
f) (1 point) The estimate for $R$ given in the introduction is not very reliable. Scientists have therefore started speculating on a better and more direct method for determining $R$. One idea for this goes as follows. Assume that we can identify and measure the Doppler shifted wavelengths $\lambda_{1}$ and $\lambda_{2}$ of radiation from the two ejected objects, corresponding to the same known original wavelength $\lambda_{0}$ in the rest frames of the objects.

Starting from the equations for the relativistic Doppler shift, $\lambda=\lambda_{0}(1-\beta \cos \phi)\left(1-\beta^{2}\right)^{-1 / 2}$, and assuming, as before, that both objects have the same speed, $v$, show that the unknown $\beta=v / c$ can be expressed in terms of $\lambda_{0}, \lambda_{1}$, and $\lambda_{2}$ as

$$
\begin{equation*}
\beta=\sqrt{1-\frac{\alpha \lambda_{0}^{2}}{\left(\lambda_{1}+\lambda_{2}\right)^{2}}} \tag{3.1}
\end{equation*}
$$

Write the numerical value of the coefficient $\alpha$ in the designated field on the answer sheet.
You may note that this means that the suggested wavelength measurements will in practice provide a new estimate of the distance.

### 3.2 Solution

a) On Figure 3.1 we mark the centers of the sources as neatly as we can. Let $\theta_{1}(t)$ be the angular distance of the left center from the cross as a function of time and $\theta_{2}(t)$ the angular distance of the right center. We measure these quantities on the figure at the given times by a ruler and convert to arcseconds according to the given scale. This results in the following numerical data:

| time <br> [days] | $\theta_{1}$ <br> [as] | $\theta_{2}$ <br> [as] |
| :---: | :---: | :---: |
| 0 | 0.139 | 0.076 |
| 7 | 0.253 | 0.139 |
| 13 | 0.354 | 0.190 |
| 20 | 0.468 | 0.253 |
| 27 | 0.601 | 0.316 |
| 34 | 0.709 | 0.367 |

The uncertainty in the readings by the ruler is estimated to be $\pm 0.5 \mathrm{~mm}$, resulting in the uncertainty of $\pm 0.013$ as in the $\theta$ values. We plot the data in Figure 3.3.


Figure 3.3: The angular distances $\theta_{1}$ and $\theta_{2}$ (in as) as functions of the time in days.
Fitting straight lines through the data results in:

$$
\begin{align*}
\boldsymbol{\omega}_{\mathbf{1}} & =\mathrm{d} \theta_{1} / \mathrm{d} t=(\mathbf{1 7 . 0} \pm \mathbf{1 . 0}) \mathbf{m a s} / \mathrm{day}=9.54 \cdot 10^{-13} \mathrm{rad} / \mathrm{s}  \tag{3.2}\\
\boldsymbol{\omega}_{\mathbf{2}} & =\mathrm{d} \theta_{2} / \mathrm{d} t=(8 . \mathbf{7} \pm \mathbf{1 . 0}) \mathbf{\mathrm { mas }} / \text { day }=4.88 \cdot 10^{-13} \mathrm{rad} / \mathrm{s}  \tag{3.3}\\
\boldsymbol{v}_{\mathbf{1}, \perp}^{\prime} & =\omega_{1} R=9.54 \cdot 10^{-13} \cdot 12.5 \cdot 3.09 \cdot 10^{19}  \tag{3.4}\\
& =\mathbf{3 . 6 8} \cdot \mathbf{1 0}^{\mathbf{8}} \mathbf{m} / \mathrm{s} \approx(\mathbf{1 . 2 3} \pm \mathbf{0 . 0 7}) \boldsymbol{c}  \tag{3.5}\\
\boldsymbol{v}_{\mathbf{2}, \perp}^{\prime} & =\mathbf{1 . 8 9} \cdot \mathbf{1 0}^{\mathbf{8}} \mathbf{m} / \mathrm{s} \approx(\mathbf{0 . 6 3} \pm \mathbf{0 . 0 7}) \boldsymbol{c} \tag{3.6}
\end{align*}
$$

b) We consider the motion of the source during the time interval $\Delta t$ from the point $A$ to the point $A^{\prime}$, see Figure 3.4.

We then have

$$
\begin{equation*}
\vec{r}_{A A^{\prime}}=\vec{r}_{A^{\prime}}-\vec{r}_{A}=\vec{v} \cdot \Delta t \tag{3.7}
\end{equation*}
$$

Now let $\Delta t^{\prime}$ denote the difference in arrival times at $O$ of the signals from $A$ and $A^{\prime}$. Due to the different distances to $A$ and $A^{\prime}$ and the finite speed of light, c , we have


Figure 3.4: The observer is at $O$ and the original position of the source is at $A$. The velocity vector is $\vec{v}$.

$$
\begin{equation*}
\Delta t^{\prime}=\Delta t+\left(r_{A^{\prime}}-r_{A}\right) / c . \tag{3.8}
\end{equation*}
$$

For small $\Delta t$, such that $v \Delta t \ll r_{A}=R$, we have

$$
\begin{equation*}
r_{A^{\prime}}-r_{A} \approx-v \Delta t \cos \phi \tag{3.9}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\Delta t^{\prime} \approx \Delta t(1-\beta \cos \phi) ; \beta=v / c \tag{3.10}
\end{equation*}
$$

This implies that an observer at $O$ will find the apparent transverse speed of the source to be

$$
\begin{equation*}
\boldsymbol{v}_{\perp}^{\prime}=\frac{\Delta x}{\Delta t^{\prime}}=\frac{\Delta x}{\Delta t(1-\beta \cos \phi)}=\frac{\boldsymbol{c} \boldsymbol{\beta} \sin \boldsymbol{\phi}}{\mathbf{1}-\boldsymbol{\beta} \cos \boldsymbol{\phi}} \tag{3.11}
\end{equation*}
$$

where we have used that the real transverse speed in the reference frame of the observer is $v_{\perp}=\Delta x / \Delta t=c \beta \sin \phi$.

The angular speed observed at $O$ is

$$
\begin{equation*}
\boldsymbol{\omega}=\frac{v_{\perp}^{\prime}}{R}=\frac{\boldsymbol{c} \boldsymbol{\beta} \sin \phi}{\boldsymbol{R}(1-\boldsymbol{\beta} \cos \phi)} \tag{3.12}
\end{equation*}
$$

c) Figure 3.5 shows the situation in this case. Note the relations given in the caption. Taking $\phi=\phi_{1}$ we have $\sin \phi_{2}=\sin \phi$ and $\cos \phi_{2}=-\cos \phi$. Equation (3.12) then gives:

$$
\begin{align*}
\omega_{1} & =\frac{\beta c \sin \phi}{R(1-\beta \cos \phi)}  \tag{3.13}\\
\omega_{2} & =\frac{\beta c \sin \phi}{R(1+\beta \cos \phi)} . \tag{3.14}
\end{align*}
$$



Figure 3.5: If the two objects have equal speeds but opposite velocities we have $v_{1}=v_{2}=$ $v, \beta_{1}=\beta_{2}=\beta$ and $\phi_{2}=\pi-\phi_{1}$.

The quantities $\omega_{1}, \omega_{2}$ and $R$ are given, but $\beta$ and $\phi$ are to be determined as stated in the problem text. Simple algebra gives:

$$
\begin{align*}
& (1-\beta \cos \phi) \omega_{1} \omega_{2}=\beta c \sin \phi \omega_{2} / R  \tag{3.15}\\
& (1+\beta \cos \phi) \omega_{2} \omega_{1}=\beta c \sin \phi \omega_{1} / R \tag{3.16}
\end{align*}
$$

Subtracting (3.15) from (3.16) gives:

$$
\begin{gather*}
2 \beta \cos \phi \omega_{2} \omega_{1}=\beta c \sin \phi\left(\omega_{1}-\omega_{2}\right) / R  \tag{3.17}\\
\tan \phi=\frac{2 R \omega_{2} \omega_{1}}{c\left(\omega_{1}-\omega_{2}\right)}  \tag{3.18}\\
\phi=\arctan \left(\frac{2 \boldsymbol{R} \omega_{\mathbf{2}} \boldsymbol{\omega}_{\mathbf{1}}}{\boldsymbol{c}\left(\boldsymbol{\omega}_{\mathbf{1}}-\boldsymbol{\omega}_{\mathbf{2}}\right)}\right) \tag{3.19}
\end{gather*}
$$

Dividing (3.15) by (3.16) gives $\beta$ in terms of $\cos \phi$ and the known quantities $\omega_{1}$ and $\omega_{2}$ :

$$
\begin{gather*}
\omega_{1}(1-\beta \cos \phi)=\omega_{2}(1+\beta \cos \phi)  \tag{3.20}\\
\boldsymbol{\beta}=\frac{\omega_{1}-\omega_{2}}{\cos \phi\left(\omega_{1}+\omega_{2}\right)} \tag{3.21}
\end{gather*}
$$

Inserting the values of $\omega_{1}$ and $\omega_{2}$ from part (a) and the given values of $R$ and $c$ we get:

$$
\begin{align*}
& \phi=\arctan (2.57)=1.20 \mathrm{rad}=68.8^{\circ} \pm 2^{\circ}  \tag{3.22}\\
& \beta=0.892 \pm 0.08 \tag{3.23}
\end{align*}
$$

d) Equation (3.11) shows that the observer will find the apparent transverse speed to be larger than or equal to the speed of light if and only if:

$$
\begin{equation*}
\frac{\beta \sin \phi}{1-\beta \cos \phi} \geq 1 \tag{3.24}
\end{equation*}
$$

If $\beta<1$ condition (3.24) is equivalent to:

$$
\begin{align*}
\beta \sin \phi & \geq 1-\beta \cos \phi  \tag{3.25}\\
\beta(\sin \phi+\cos \phi) & \geq 1  \tag{3.26}\\
\beta \sqrt{2}\left(\sin \phi \cos \frac{\pi}{4}+\cos \phi \sin \frac{\pi}{4}\right) & \geq 1  \tag{3.27}\\
\sin \left(\phi+\frac{\pi}{4}\right) & \geq \frac{1}{\beta \sqrt{2}} \tag{3.28}
\end{align*}
$$

and hence (3.24) is satisfied if:

$$
\begin{equation*}
\beta>f(\phi)=(\sqrt{2} \sin (\phi+\pi / 4))^{-1} \tag{3.29}
\end{equation*}
$$

The physically relevant region in the $(\beta, \phi)$-plane is:

$$
\begin{equation*}
(\beta, \phi) \in[0,1[\times[0, \pi] . \tag{3.30}
\end{equation*}
$$

It is obvious that (3.24) can only be satisfied for $\phi \in[0, \pi / 2]$ and (3.28) can only have a solution for $\phi$ if $\beta \geq 1 / \sqrt{2}$.

We therefore take a closer look at the region

$$
\begin{equation*}
(\beta, \phi) \in\left[2^{-1 / 2}, 1[\times[0, \pi / 2]\right. \tag{3.31}
\end{equation*}
$$

The mapping

$$
\begin{equation*}
(\beta, \phi) \mapsto \beta \sin \left(\phi+\frac{\pi}{4}\right) \tag{3.32}
\end{equation*}
$$

is continuous in this region. It is therefor sufficient to look at the boundary of the region, defined by the equality sign in (3.28):

$$
\begin{equation*}
\beta \sin \left(\phi+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \tag{3.33}
\end{equation*}
$$

This defines $\beta$ as a function of $\phi$ which is shown in Figure 3.6 as the curve bounding the shaded area where $v_{\perp}^{\prime}>c$.
e) To find the extrema of $v_{\perp}^{\prime}$ as a function of $\phi$ we differentiate (3.11) and get

$$
\begin{equation*}
\frac{d}{d \phi}\left(\frac{v_{\perp}^{\prime}}{c}\right)=\frac{\beta(\cos \phi-\beta)}{(1-\beta \cos \phi)^{2}} . \tag{3.34}
\end{equation*}
$$

This is zero for $\phi=\phi_{m}$ where:


Figure 3.6: The region between the horizontal line and the curve in the upper left hand corner shows where $v_{\perp}^{\prime} / c>1$.

Figure 3.7: The curved surface is $v_{\perp}^{\prime} / c$ as a function of $\beta$ and $\phi$. The plane represents the constant function $\beta=1$.

$$
\begin{equation*}
\left.\left.\cos \phi_{m}=\beta ; \phi_{m}=\arccos \beta \in\right] 0, \pi / 2\right] \tag{3.35}
\end{equation*}
$$

To see that this is indeed a maximum, we differentiate (3.34) again and get:

$$
\begin{equation*}
\frac{d^{2}}{d \phi^{2}}\left(\frac{v_{\perp}^{\prime}}{c}\right)=-\beta\left(\frac{\sin \phi}{(1-\beta \cos \phi)^{2}}+2 \frac{\beta \sin \phi(\cos \phi-\beta)}{(1-\beta \cos \phi)^{3}}\right) \tag{3.36}
\end{equation*}
$$

At the extremum

$$
\begin{equation*}
\left.\frac{d^{2}}{d \phi^{2}}\left(\frac{v_{\perp}^{\prime}}{c}\right)\right|_{\phi_{m}}=-\frac{\beta \sin \phi_{m}}{\left(1-\beta^{2}\right)^{2}}<0 \tag{3.37}
\end{equation*}
$$

showing that $\phi_{m}$ corresponds to a maximum. From (3.11) and (3.35) the maximum apparent transverse speed is given:

$$
\begin{equation*}
\left(v_{\perp}^{\prime}\right)_{\max }=\frac{\beta c}{\sqrt{1-\beta^{2}}} \tag{3.38}
\end{equation*}
$$

From this and (3.35) we see that

$$
\begin{equation*}
\left(v_{\perp}^{\prime}\right)_{\max } \underset{\beta \rightarrow 1}{\longrightarrow} \infty ; \phi_{m} \underset{\beta \rightarrow 1}{\longrightarrow} 0 . \tag{3.39}
\end{equation*}
$$

Figure 3.7 shows $v_{\perp}^{\prime} / c$ as a function of $\beta$ and $\phi$ in the region $(\beta, \phi) \in\left[2^{-1 / 2}, 1[\times[0, \pi / 2]\right.$.
f) We have the equations for relativistic Doppler-shift:

$$
\begin{equation*}
\frac{\lambda_{1,2}}{\lambda_{0}}=\frac{1 \mp \beta \cos \phi}{\sqrt{1-\beta^{2}}} \tag{3.40}
\end{equation*}
$$

We add them, define an auxiliary ratio $\rho$ and solve for $\beta$.

$$
\begin{gather*}
\rho:=\frac{\lambda_{1}+\lambda_{2}}{2 \lambda_{0}}=\frac{1}{\sqrt{1-\beta^{2}}}  \tag{3.41}\\
\rho^{2}\left(1-\beta^{2}\right)=1  \tag{3.42}\\
\beta=\sqrt{1-1 / \rho^{2}}=\sqrt{1-\frac{4 \lambda_{0}^{2}}{\left(\lambda_{1}+\lambda_{2}\right)^{2}}} \tag{3.43}
\end{gather*}
$$

giving

$$
\begin{equation*}
\alpha=4 \tag{3.44}
\end{equation*}
$$

Adding equation (3.43) to the set of equations (3.18) and (3.21) we have three equations which can be solved for the three unknowns $\beta, \phi$ and $R$. For instance, we may calculate $\beta$ from (3.43), insert that into (3.21), and solve for $\phi$. The distance $R$ can then be obtained from (3.18). Thus the measurement of the Doppler-shifted wavelengths turns out to give an estimate of the distance to the source provided that $\omega_{1}$ and $\omega_{2}$ are known.

### 3.3 Grading scheme

| Part 1(a) |  |
| :---: | :---: |
| Answer i): equation (3.2), $\omega_{1}$ in the range (16.5-17.5) mas/day | 0.8 |
| Answer ii): equation (3.3), $\omega_{2}$ in the range (8.2-9.2) mas/day | 0.8 |
| Answer iii): equation (3.4), for $v_{1, \perp}^{\prime}$ in the range (1.13-1.30)c | 0.2 |
| Answer iv): equation (3.6), for $v_{2, \perp}^{\prime}$ in the range (0.56-0.70)c | 0.2 |
| Part 1(b) |  |
| Answer i): $v_{\perp}^{\prime}(\beta, \phi)$, equation (3.11) | 2.5 |
| Answer ii): $\omega(\beta, \phi)$, equation (3.12) | 0.5 |
| Part 1(c) |  |
| Answer i): $\phi\left(\omega_{1}, \omega_{2}\right)$, equation (3.19) | 0.3 |
| Answer ii): $\beta\left(\omega_{1}, \omega_{2}\right)$, equation (3.21) | 0.3 |
| Answer iii): $\phi$ numerical in the range $67^{\circ}-71^{\circ}$ | 0.2 |
| Answer iv): $\beta$ numerical in the range 0.81-0.97 | 0.2 |
| Part 1(d) |  |
| Answer i): Condition $\beta>f(\phi)$, equation (3.29) | 1.0 |
| Answer ii): Condition on ( $\beta, \phi$ ), graph | 1.0 |
| Part 1(e) |  |
| Answer: $\left(v_{\perp}^{\prime}\right)_{\max }$, equation (3.38) | 1.0 |
| Part 1(f) |  |
| Answer: $\beta$ in terms of $\lambda$-s, by $\alpha$, equation (3.44) | 1.0 |

# $29^{\text {th }}$ International Physics Olympiad 

Reykjavik, Iceland

## Experimental competition

Monday, July 6th, 1998

## Time available: 5 hours

## Read this first:

Use only the pen provided.

1. Use only the front side of the answer sheets and paper.
2. Use as little text as possible in your answers; express yourself primarily with equations, numbers and figures. Summarize your results on the answer sheets.
3. Please indicate on all sheets the name of your team, your student number, the page number and the total number of pages.
4. At the end of the exam please put your answer sheets, pages and graphs in numerical order and leave them on your table.
5. Use of a calculator is allowed.

This set of problems consists of $\mathbf{6}$ pages.

Examination prepared at: University of Iceland, Department of Physics.

## Instrumentation provided:

A Platform with 6 banana jacks
B Pickup coil embedded into the platform

C Ferrite U-core with two coils marked " A " and ${ }^{`} \mathrm{~B}$ "

D Ferrite U-core without coils
E Aluminium foils of thicknesses: $50 \mu \mathrm{~m}, 100 \mu \mathrm{~m}$ and $200 \mu \mathrm{~m}$

F Function generator with output leads

G Two multimeters
H Six leads with banana plugs
I Two rubber bands and two
 plastic spacers

## Multimeters

The multimeters are two-terminal devices that in this experiment are used for measuring AC voltages, AC currents, frequency and resistance. In all cases one of the terminals is the one marked $\mathbf{C O M}$. For the voltage, frequency and resistance measurements the other terminal is the red one marked $\mathbf{V}-\boldsymbol{\Omega}$. For current measurements the other terminal is the yellow one marked $\mathbf{m A}$. With the central dial you select the meter function ( $\mathrm{V} \sim$ for AC voltage, $\mathrm{A} \sim$ for AC current, Hz for frequency and $\Omega$ for resistance) and the measurement range. For the AC modes the measurement uncertainty is $\pm$ ( $4 \%$ of reading +10 units of the last digit). To get accurate current measurements a change of range is recommended if the reading is less than $10 \%$ of full scale.

## Function generator

To turn on the generator you press in the red button marked PWR. Select the 10 kHz range by pressing the button marked 10k, and select the sine waveform by pressing the second button from the right marked with a wave symbol. No other buttons should be selected. You can safely turn the amplitude knob fully clockwise. The frequency is selected with the large dial on the left. The dial reading multiplied by the range selection gives the output frequency. The frequency can be verified at any time with one of the multimeters. Use the output marked MAIN, which has $50 \Omega$ internal resistance.

## Ferrite cores

Handle the ferrite cores gently, they are brittle!! Ferrite is a ceramic magnetic material, with low electrical conductivity. Eddy current losses in the cores are therefore low.

## Banana jacks

To connect a coil lead to a banana jack, you loosen the colored plastic nut, place the tinned end between the metal nut and plastic nut, and tighten it again.


Figure 1: Experimental arrangement for part I.

## Part I. Magnetic shielding with eddy currents

Time-dependent magnetic fields induce eddy currents in conductors. The eddy currents in turn produce a counteracting magnetic field. In superconductors the induced eddy currents will expel the magnetic field completely from the interior of the conductor. Due to the finite conductivity of normal metals they are not as effective in shielding magnetic fields.
To describe the shielding effect of aluminium foils we will apply a phenomenological model

$$
\begin{equation*}
B=B_{0} e^{-\alpha d} \tag{1}
\end{equation*}
$$

where $B_{0}$ is the magnetic field in the absence of foils. $B$ is the magnetic field beneath the foils, $\alpha$ an attenuation constant, and $d$ the foil thickness.

## Experiment

Put the ferrite core with the coils, with legs down, on the raised block such that coil A is directly above the pickup coil embedded in the platform, as shown in Figure 1. Secure the core on the block by stretching the rubber bands over the core and under the block recess.

1. Connect the leads for coils A and B to the jacks. Measure the resistance of all coils to make sure you have good connections. You should expect values of less than $10 \Omega$. Write your values in field 1 on the answer sheet.
2. Collect data to validate the model above and evaluate the attenuation constant $\alpha$ for the aluminum foils ( $50-300 \mu \mathrm{~m}$ ), for frequencies in the range of $5-20 \mathrm{kHz}$. Place the foils inside the square, above the pickup coil, and apply a sinusoidal voltage to coil A. Write your results in field 2 on the answer sheet.
3. Plot $\alpha$ versus frequency, and write in field 3 on the answer sheet, an expression describing the function $\alpha(f)$.

## Part II. Magnetic flux linkage

The response of two coils on a closed ferrite core to an external alternating voltage ( $V_{g}$ ) from a sinusoidal signal generator is studied.

## Theory

In the following basic theoretical discussion, and in the treatment of the data, it is assumed that the ohmic resistance in the two coils and hysteresis losses in the core have insignificant influence on the measured currents and voltages. Because of these simplifications in the treatment below, some deviations will occur between measured and calculated values.

## Single coil

Let us first look at a core with a single coil, carrying a current $I$. The magnetic flux $\Phi$, that the current creates in the ferrite core inside the coil, is proportional to the current $I$ and to the number of windings $N$. The flux depends furthermore on a geometrical factor $g$, which is determined by the size and shape of the core, and the magnetic permeability $\mu=\mu_{r} \mu_{0}$, which describes the magnetic properties of the core material. The relative permeability is denoted $\mu_{r}$ and $\mu_{0}$ is the permeability of free space. The magnetic flux $\Phi$ is thus given by

$$
\begin{equation*}
\Phi=\mu g N I=c N I \tag{2}
\end{equation*}
$$

where $c=\mu \mathrm{g}$. The induced voltage is given by Faraday's law of induction,

$$
\begin{equation*}
\varepsilon(t)=-N \frac{d \Phi(t)}{d t}=-c N^{2} \frac{d I(t)}{d t} \tag{3}
\end{equation*}
$$

The conventional way to describe the relationship between current and voltage for a coil is through the self inductance of the coil $L$, defined by,

$$
\begin{equation*}
\varepsilon(t)=-L \frac{d I(t)}{d t} \tag{4}
\end{equation*}
$$

A sinusoidal signal generator connected to the coil will drive a current through it given by

$$
\begin{equation*}
I(t)=I_{0} \sin \omega t \tag{5}
\end{equation*}
$$

where $\omega$ is the angular frequency and $I_{0}$ is the amplitude of the current. As follows from equation (3) this alternating current will induce a voltage across the coil given by

$$
\begin{equation*}
\varepsilon(t)=-\omega c N^{2} I_{0} \cos \omega t \tag{6}
\end{equation*}
$$

The current will be such that the induced voltage is equal to the signal generator voltage $V_{g}$. There is a 90 degree phase difference between the current and the voltage. If we only look at the magnitudes of the alternating voltage and current, allowing for this phase difference, we have

$$
\begin{equation*}
\varepsilon=\omega c N^{2} I \tag{7}
\end{equation*}
$$

## Two coils

Let us now assume that we have two coils on one core. Ferrite cores can be used to link magnetic flux between coils. In an ideal core the flux will be the same for all cross sections of the core. Due to flux leakage in real cores a second coil on the core will in general experience a reduced flux compared to the
flux-generating coil. The flux $\Phi_{B}$ in the secondary coil B is therefore related to the flux $\Phi_{A}$ in the primary coil A through

$$
\begin{equation*}
\Phi_{B}=k \Phi_{A} \tag{8}
\end{equation*}
$$

Similarly a flux component $\Phi_{B}$ created by a current in coil B will create a flux $\Phi_{A}=k \Phi_{B} \quad$ in coil A. The factor $k$, which is called the coupling factor, has a value less than one.
The ferrite core under study has two coils A and B in a transformer arrangement. Let us assume that coil A is the primary coil (connected to the signal generator). If no current flows in coil B ( $I_{B}=0$ ), the induced voltage $\varepsilon_{A}$ due to $I_{A}$ is equal and opposite to $V_{g}$. The flux created by $I_{A}$ inside the secondary coil is determined by equation (8) and the induced voltage in coil B is

$$
\begin{equation*}
\varepsilon_{B}=\omega k c N_{A} N_{B} I_{A} \tag{9}
\end{equation*}
$$

If a current $I_{B}$ flows in coil B, it will induce a voltage in coil A which is described by a similar expression. The total voltage across the coil A will then be given by

$$
\begin{equation*}
V_{g}=\varepsilon_{A}=\omega c N_{A}^{2} I_{A}-\omega k c N_{A} N_{B} I_{B} \tag{10}
\end{equation*}
$$

The current in the secondary coil thus induces an opposing voltage in the primary coil, leading to an increase in $I_{A}$. A similar equation can be written for $\varepsilon_{B}$. As can be verified by measurements, $k$ is independent of which coil is taken as the primary coil.

## Experiment

Place the two U-cores together as shown in Figure 2, and fasten them with the rubber bands. Set the function generator to produce a 10 kHz , sine wave. Remember to set the multimeters to the most sensitive range suitable for each measurement. The numbers of the windings of the two coils, A and B, are: $N_{A}=$ 150 turns and $\quad N_{B}=100$ turns ( $\pm 1$ turn on each coil).


Figure 2: A transformer with a closed magnetic circuit.

1. Derive algebraic expressions for the self inductances $L_{A}$ and $L_{B}$, and the coupling factor $k$, in terms of measured and given quantities and write your results in field 1.a on the answer sheet. Draw circuit diagrams in field 1.b on the answer sheet, showing how these quantities are determined. Calculate the numerical values of $L_{A} \quad, L_{B} \quad$ and $k$ and write their values in field 1.c on the answer sheet.
2. When the secondary coil is short-circuited, the current $I_{P}$ in the primary coil will increase. Use the equations above to derive an expression giving $I_{P}$ explicitly and write your result in field 2.a on the answer sheet. Measure $I_{P}$ and write your value in field 2.b on the answer sheet.
3. Coils A and B can be connected in series in two different ways such that the two flux contributions are either added to or subtracted from each other.
3.1. Find the self inductance of the serially connected coils, $L_{A+B}$, from measured quantities in the case where the flux contributions produced by the current $I$ in the two coils add to (strengthen) each other and write your answer in field 3.1 on the answer sheet.
3.2. Measure the voltages $V_{A}$ and $V_{B}$ when the flux contributions of the two coils oppose each other. Write your values in field 3.2.a on the answer sheet and the ratio of the voltages in field 3.2.b. Derive an expression for the ratio of the voltages across the two coils and write it in field 3.2.c on the answer sheet.
4. Use the results obtained to verify that the self inductance of a coil is proportional to the square of the number of its windings and write your result in field 4 on the answer sheet.
5. Verify that it was justified to neglect the resistances of the coils and write your arguments as mathematical expressions in field 5 on the answer sheet.
6. Thin plastic spacers inserted between the two half cores (as shown in Figure 3) reduce the coil inductances drastically. Use this reduction to determine the relative permeability $\mu_{r}$ of the ferrite material, given Ampere's law and continuity of the magnetic field $\mathbf{B}$ across the ferrite - plastic interface.

Assume $\mu=\mu_{0}=4 \pi \times 10^{-7} \mathrm{Ns}^{2} / \mathrm{C}^{2}$ for the plastic spacers and a spacer thickness of 1.6 mm . The geometrical factor can be determined from Ampere's law

$$
\begin{equation*}
\oint \frac{1}{\mu} B d l=I_{\text {total }} \tag{11}
\end{equation*}
$$

where $I_{\text {total }}$ is the total current flowing through a surface bounded by the integration path. Write your algebraic expression for $\mu_{r}$ in field 6.a on the answer sheet and your numerical value in field 6.b.


Figure 3: The ferrite cores with the two spacers in place.

# 30th International Physics Olympiad 

## Padua, Italy

# Theoretical competition 

Thursday, July 22nd, 1999

## Please read this first:

1. The time available is 5 hours for 3 problems.
2. Use only the pen provided.
3. Use only the front side of the provided sheets.
4. In addition to the problem texts, that contain the specific data for each problem, a sheet is provided containing a number of general physical constants that may be useful for the problem solutions.
5. Each problem should be answered on separate sheets.
6. In addition to "blank" sheets where you may write freely, for each problem there is an Answer sheet where you must summarize the results you have obtained. Numerical results must be written with as many digits as appropriate to the given data; don't forget the units.
7. Please write on the "blank" sheets whatever you deem important for the solution of the problem, that you wish to be evaluated during the marking process. However, you should use mainly equations, numbers, symbols, figures, and use as little text as possible.
8. It's absolutely imperative that you write on top of each sheet that you'll use: your name ("NAME"), your country ("TEAM"), your student code (as shown on the identification tag, "CODE"), and additionally on the "blank" sheets: the problem number ("Problem"), the progressive number of each sheet (from 1 to $N$, "Page n.") and the total number ( $N$ ) of "blank" sheets that you use and wish to be evaluated for that problem ("Page total"). It is also useful to write the section you are answering at the beginning of each such section. If you use some sheets for notes that you don't wish to be evaluated by the marking team, just put a large cross through the whole sheet, and don't number it.
9. When you've finished, turn in all sheets in proper order (for each problem: answer sheet first, then used sheets in order; unused sheets and problem text at the bottom) and put them all inside the envelope where you found them; then leave everything on your desk. You are not allowed to take any sheets out of the room.

## This set of problems consists of 13 pages (including this one, the answer sheets and the page with the physical constants)

[^7]
## Absorption of radiation by a gas

A cylindrical vessel, with its axis vertical, contains a molecular gas at thermodynamic equilibrium. The upper base of the cylinder can be displaced freely and is made out of a glass plate; let's assume that there is no gas leakage and that the friction between glass plate and cylinder walls is just sufficient to damp oscillations but doesn't involve any significant loss of energy with respect to the other energies involved. Initially the gas temperature is equal to that of the surrounding environment. The gas can be considered as perfect within a good approximation. Let's assume that the cylinder walls (including the bases) have a very low thermal conductivity and capacity, and therefore the heat transfer between gas and environment is very slow, and can be neglected in the solution of this problem.

Through the glass plate we send into the cylinder the light emitted by a constant power laser; this radiation is easily transmitted by air and glass but is completely absorbed by the gas inside the vessel. By absorbing this radiation the molecules reach excited states, where they quickly emit infrared radiation returning in steps to the molecular ground state; this infrared radiation, however, is further absorbed by other molecules and is reflected by the vessel walls, including the glass plate. The energy absorbed from the laser is therefore transferred in a very short time into thermal movement (molecular chaos) and thereafter stays in the gas for a sufficiently long time.

We observe that the glass plate moves upwards; after a certain irradiation time we switch the laser off and we measure this displacement.

1. Using the data below and - if necessary - those on the sheet with physical constants, compute the temperature and the pressure of the gas after the irradiation. [2 points]
2. Compute the mechanical work carried out by the gas as a consequence of the radiation absorption. [1 point]
3. Compute the radiant energy absorbed during the irradiation. [2 points]
4. Compute the power emitted by the laser that is absorbed by the gas, and the corresponding number of photons (and thus of elementary absorption processes) per unit time. [1.5 points]
5. Compute the efficiency of the conversion process of optical energy into a change of mechanical potential energy of the glass plate. [1 point]

Thereafter the cylinder axis is slowly rotated by $90^{\circ}$, bringing it into a horizontal direction. The heat exchanges between gas and vessel can still be neglected.
6. State whether the pressure and/or the temperature of the gas change as a consequence of such a rotation, and - if that is the case - what is its/their new value. [2.5 points]

## Data

Room pressure: $p_{0}=101.3 \mathrm{kPa}$
Room temperature: $T_{0}=20.0^{\circ} \mathrm{C}$
Inner diameter of the cylinder: $2 r=100 \mathrm{~mm}$
Mass of the glass plate: $m=800 \mathrm{~g}$
Quantity of gas within the vessel: $n=0.100 \mathrm{~mol}$
Molar specific heat at constant volume of the gas: $c_{\mathrm{V}}=20.8 \mathrm{~J} /(\mathrm{mol} \cdot \mathrm{K})$
Emission wavelength of the laser: $\lambda=514 \mathrm{~nm}$
Irradiation time: $\Delta t=10.0 \mathrm{~s}$
Displacement of the movable plate after irradiation: $\Delta s=30.0 \mathrm{~mm}$
$\qquad$
$\qquad$

| Problem | 1 |
| :---: | :---: |
| Page n. | A |
| Page total |  |

## Answer sheet

In this problem you are requested to give your results both as analytical expressions and with numerical data and units: write expressions first and then data (e.g. $A=b c=1.23 \mathrm{~m}^{2}$ ).

1. Gas temperature after the irradiation $\qquad$
Gas pressure after the irradiation $\qquad$
2. Mechanical work carried out $\qquad$
3. Overall optical energy absorbed by the gas $\qquad$
4. Optical laser power absorbed by the gas $\qquad$
Absorption rate of photons (number of absorbed photons per unit time) $\qquad$
5. Efficiency in the conversion of optical energy into change of mechanical potential energy of the glass plate $\qquad$
6. Owing to the cylinder rotation, is there a pressure change? YES $\square$ NO $\square$

If yes, what is its new value?
Owing to the cylinder rotation, is there a temperature change? YES $\square$NO

If yes, what is its new value? $\qquad$

## Physical constants and general data

In addition to the numerical data given within the text of the individual problems, the knowledge of some general data and physical constants may be useful, and you may find them among the following ones. These are nearly the most accurate data currently available, and they have thus a large number of digits; you are expected, however, to write your results with a number of digits that must be appropriate for each problem.

Speed of light in vacuum: $c=299792458 \mathrm{~m} \cdot \mathrm{~s}^{-1}$
Magnetic permeability of vacuum: $\mu_{0}=4 \pi \cdot 10^{-7} \mathrm{H} \cdot \mathrm{m}^{-1}$
Dielectric constant of vacuum: $\varepsilon_{0}=8.8541878 \mathrm{pF} \cdot \mathrm{m}^{-1}$
Gravitational constant: $G=6.67259 \cdot 10^{-11} \mathrm{~m}^{3} /\left(\mathrm{kg} \cdot \mathrm{s}^{2}\right)$
Gas constant: $R=8.314510 \mathrm{~J} /(\mathrm{mol} \cdot \mathrm{K})$
Boltzmann's constant: $k=1.380658 \cdot 10^{-23} \mathrm{~J} \cdot \mathrm{~K}^{-1}$
Stefan's constant: $\sigma=56.703 \mathrm{nW} /\left(\mathrm{m}^{2} \cdot \mathrm{~K}^{4}\right)$
Proton charge: $e=1.60217733 \cdot 10^{-19} \mathrm{C}$
Electron mass: $m_{\mathrm{e}}=9.1093897 \cdot 10^{-31} \mathrm{~kg}$
Planck's constant: $h=6.6260755 \cdot 10^{-34} \mathrm{~J} \cdot \mathrm{~s}$
Base of centigrade scale: $T_{\mathrm{K}}=273.15 \mathrm{~K}$
Sun mass: $M_{\mathrm{S}}=1.991 \cdot 10^{30} \mathrm{~kg}$
Earth mass: $M_{\mathrm{E}}=5.979 \cdot 10^{24} \mathrm{~kg}$
Mean radius of Earth: $r_{\mathrm{E}}=6.373 \mathrm{Mm}$
Major semiaxis of Earth orbit: $R_{\mathrm{E}}=1.4957 \cdot 10^{11} \mathrm{~m}$
Sidereal day: $d_{\mathrm{S}}=86.16406 \mathrm{ks}$
Year: $y=31.558150 \mathrm{Ms}$
Standard value of the gravitational field at the Earth surface: $g=9.80665 \mathrm{~m} \cdot \mathrm{~s}^{-2}$
Standard value of the atmospheric pressure at sea level: $p_{0}=101325 \mathrm{~Pa}$
Refractive index of air for visibile light, at standard pressure and $15^{\circ} \mathrm{C}$ : $n_{\text {air }}=1.000277$
Solar constant: $S=1355 \mathrm{~W} \cdot \mathrm{~m}^{-2}$
Jupiter mass: $M=1.901 \cdot 10^{27} \mathrm{~kg}$
Equatorial Jupiter radius: $R_{\mathrm{B}}=69.8 \mathrm{Mm}$
Average radius of Jupiter's orbit: $R=7.783 \cdot 10^{11} \mathrm{~m}$
Jovian day: $d_{\mathrm{J}}=35.6 \mathrm{ks}$
Jovian year: $y_{\mathrm{J}}=374.32$ Ms
$\pi$ : 3.14159265

## Magnetic field with a V-shaped wire

Among the first successes of the interpretation by Ampère of magnetic phenomena, we have the computation of the magnetic field $\mathbf{B}$ generated by wires carrying an electric current, as compared to early assumptions originally made by Biot and Savart.

A particularly interesting case is that of a very long thin wire, carrying a constant current $i$, made out of two rectilinear sections and bent in the form of a " V ", with angular half-span ${ }^{1} \alpha$ (see figure). According to Ampère's computations, the magnitude $B$ of the magnetic field in a given point $P$ lying on the axis of the " V ", outside of it and at a distance $d$ from its vertex, is proportional to $\tan \left(\frac{\alpha}{2}\right)$. Ampère's work was later embodied in Maxwell's electromagnetic theory, and is universally accepted.


Using our contemporary knowledge of electromagnetism,

1. Find the direction of the field $\mathbf{B}$ in $\mathbf{P}$. [1 point]
2. Knowing that the field is proportional to $\tan \left(\frac{\alpha}{2}\right)$, find the proportionality factor $k$ in $|\mathbf{B}(\mathrm{P})|=k \tan \left(\frac{\alpha}{2}\right) . \quad[1.5$ points]
3. Compute the field $\mathbf{B}$ in a point $\mathrm{P}^{*}$ symmetric to P with respect to the vertex, i.e. along the axis and at the same distance $d$, but inside the " V " (see figure). [2 points]

[^8]
4. In order to measure the magnetic field, we place in P a small magnetic needle with moment of inertia $I$ and magnetic dipole moment $\mu$; it oscillates around a fixed point in a plane containing the direction of $\mathbf{B}$. Compute the period of small oscillations of this needle as a function of $B$. [2.5 points]

In the same conditions Biot and Savart had instead assumed that the magnetic field in P might have been (we use here the modern notation) $B(P)=\frac{i \mu_{0} \alpha}{\pi^{2} d}$, where $\mu_{0}$ is the magnetic permeability of vacuum. In fact they attempted to decide with an experiment between the two interpretations (Ampère's and Biot and Savart's) by measuring the oscillation period of the magnetic needle as a function of the "V" span. For some $\alpha$ values, however, the differences are too small to be easily measurable.
5. If, in order to distinguish experimentally between the two predictions for the magnetic needle oscillation period $T$ in P , we need a difference by at least $10 \%$, namely $T_{1}>1.10 T_{2}$ ( $T_{1}$ being the Ampere prediction and $T_{2}$ the Biot-Savart prediction) state in which range, approximately, we must choose the "V" half-span $\alpha$ for being able to decide between the two interpretations. [3 points]

## Hint

Depending on which path you follow in your solution, the following trigonometric equation might be useful: $\tan \left(\frac{\alpha}{2}\right)=\frac{\sin \alpha}{1+\cos \alpha}$
$\qquad$
$\qquad$

| Problem | 2 |
| :---: | :---: |
| Page n. | A |
| Page total |  |

## Answer sheet

In this problem write the requested results as analytic expressions, not as numerical values and units, unless explicitly indicated.

1. Using the following sketch draw the direction of the $\mathbf{B}$ field (the length of the vector is not important). The sketch is a spatial perspective view.

2. Proportionality factor $k$ $\qquad$
3. Absolute value of the magnetic field intensity at the point $P^{*}$, as described in the text. $\qquad$
Draw the direction of the $\mathbf{B}$ field in the above sketch
4. Period of the small angle oscillations of the magnet
5. Write for which range of $\alpha$ values (indicating here the numerical values of the range limits) the ratio between the oscillation periods, as predicted by Ampère and by Biot and Savart, is larger than 1.10:

## Problem 3

## A space probe to Jupiter

We consider in this problem a method frequently used to accelerate space probes in the desired direction. The space probe flies by a planet, and can significantly increase its speed and modify considerably its flight direction, by taking away a very small amount of energy from the planet's orbital motion. We analyze here this effect for a space probe passing near Jupiter.

The planet Jupiter orbits around the Sun along an elliptical trajectory, that can be approximated by a circumference of average radius $R$; in order to proceed with the analysis of the physical situation we must first:

1. $\quad$ Find the speed $V$ of the planet along its orbit around the Sun. [ 1.5 points]
2. When the probe is between the Sun and Jupiter (on the segment Sun-Jupiter), find the distance from Jupiter where the Sun's gravitational attraction balances that by Jupiter. [1 point]

A space probe of mass $m=825 \mathrm{~kg}$ flies by Jupiter. For simplicity assume that the trajectory of the space probe is entirely in the plane of Jupiter's orbit; in this way we neglect the important case in which the space probe is expelled from Jupiter's orbital plane.

We only consider what happens in the region where Jupiter's attraction overwhelms all other gravitational forces.

In the reference frame of the Sun's center of mass the initial speed of the space probe is $v_{0}$ $=1.00 \cdot 10^{4} \mathrm{~m} / \mathrm{s}$ (along the positive $y$ direction) while Jupiter's speed is along the negative $x$ direction (see figure 1); by "initial speed" we mean the space probe speed when it's in the interplanetary space, still far from Jupiter but already in the region where the Sun's attraction is negligible with respect to Jupiter's. We assume that the encounter occurs in a sufficiently short time to allow neglecting the change of direction of Jupiter along its orbit around the Sun. We also assume that the probe passes behind Jupiter, i.e. the $x$ coordinate is greater for the probe than for Jupiter when the $y$ coordinate is the same.


Figure 1: View in the Sun center of mass system. O denotes Jupiter's orbit, s is the space probe.
3. Find the space probe's direction of motion (as the angle $\varphi$ between its direction and the $x$ axis) and its speed $v$ ’ in Jupiter's reference frame, when it's still far away from Jupiter. [2 points]
4. Find the value of the space probe's total mechanical energy $E$ in Jupiter's reference frame, putting - as usual - equal to zero the value of its potential energy at a very large distance, in this case when it is far enough to move with almost constant velocity owing to the smallness of all gravitational interactions. [1 point]

The space probe's trajectory in the reference frame of Jupiter is a hyperbola and its equation in polar coordinates in this reference frame is

$$
\begin{equation*}
\frac{1}{r}=\frac{G M}{v^{\prime 2} b^{2}}\left(1+\sqrt{1+\frac{2 E v^{\prime 2} b^{2}}{G^{2} M^{2} m}} \cos \theta\right) \tag{1}
\end{equation*}
$$

where $b$ is the distance between one of the asymptotes and Jupiter (the so called impact parameter), $E$ is the probe's total mechanical energy in Jupiter's reference frame, $G$ is the gravitational constant, $M$ is the mass of Jupiter, $r$ and $\theta$ are the polar coordinates (the radial distance and the polar angle).

Figure 2 shows the two branches of a hyperbola as described by equation (1); the asymptotes and the polar co-ordinates are also shown. Note that equation (1) has its origin in the "attractive focus" of the hyperbola. The space probe's trajectory is the attractive trajectory (the Final
emphasized branch).


Space Probe
Figure 2
5. Using equation (1) describing the space probe's trajectory, find the total angular deviation $\Delta \theta$ in Jupiter's reference frame (as shown in figure 2) and express it as a function of initial speed $v$ ' and impact parameter $b$. [2 points]
6. Assume that the probe cannot pass Jupiter at a distance less than three Jupiter radii from the center of the planet; find the minimum possible impact parameter and the maximum possible angular deviation. [1 point]
7. Find an equation for the final speed $v$ " of the probe in the Sun's reference frame as a function only of Jupiter's speed $V$, the probe's initial speed $v_{0}$ and the deviation angle $\Delta \theta$. [1 point]
8. Use the previous result to find the numerical value of the final speed $v$ " in the Sun's reference frame when the angular deviation has its maximum possible value.
[0.5 points]

## Hint

Depending on which path you follow in your solution, the following trigonometric formulas might be useful:

$$
\begin{aligned}
& \sin (\alpha+\beta)=\sin \alpha \cos \beta+\cos \alpha \sin \beta \\
& \cos (\alpha+\beta)=\cos \alpha \cos \beta-\sin \alpha \sin \beta
\end{aligned}
$$

NAME___
TEAM___
CODE_

| Problem | 3 |
| :---: | :---: |
| Page n. | A |
| Page total |  |

## Answer sheet

Unless explixitly requested to do otherwise, in this problem you are supposed to write your results both as analytic equations (first) and then as numerical results and units (e.g. $A=b c=1.23$ $\mathrm{m}^{2}$ ).

1. Speed $V$ of Jupiter along its orbit
2. Distance from Jupiter where the two gravitational attractions balance each other
3. Initial speed $v$ ' of the space probe in Jupiter's reference frame and the angle $\varphi$ its direction forms with the $x$ axis, as defined in figure 1, $\qquad$
4. Total energy $E$ of the space probe in Jupiter's reference frame $\qquad$
5. Write a formula linking the probe's deviation $\Delta \theta$ in Jupiter's reference frame to the impact parameter $b$, the initial speed $v$ ’ and other known or computed quantities $\qquad$
$\qquad$
6. If the distance from Jupiter's center can't be less than three Jovian radii, write the minimum impact parameter and the maximum angular deviation: $b=$ $\qquad$ $\Delta \theta=$ $\qquad$
7. Equation for the final probe speed $v$ " in the Sun's reference frame as a function of $V, v_{0}$ and $\Delta \theta$ $\qquad$
8. Numerical value of the final speed in the Sun's reference frame when the angular deviation has its maximum value as computed in step 6 $\qquad$ Final

# 30th International Physics Olympiad 

Padua, Italy<br>\section*{Experimental competition}

Tuesday, July 20th, 1999

## Before attempting to assemble your equipment, read the problem text completely!

## Please read this first:

1. The time available is 5 hours for one experiment only.
2. Use only the pen provided.
3. Use only the front side of the provided sheets.
4. In addition to "blank" sheets where you may write freely, there is a set of Answer sheets where you must summarize the results you have obtained. Numerical results must be written with as many digits as appropriate; don't forget the units. Try - whenever possible to estimate the experimental uncertainties.
5. Please write on the "blank" sheets the results of all your measurements and whatever else you deem important for the solution of the problem, that you wish to be evaluated during the marking process. However, you should use mainly equations, numbers, symbols, graphs, figures, and use as little text as possible.
6. It's absolutely imperative that you write on top of each sheet that you'll use: your name ("NAME"), your country ("TEAM"), your student code (as shown on your identification tag, "CODE"), and additionally on the "blank" sheets: the progressive number of each sheet (from 1 to $N$, "Page n.") and the total number ( $N$ ) of "blank" sheets that you use and wish to be evaluated ("Page total"); leave the "Problem" field blank. It is also useful to write the number of the section you are answering at the beginning of each such section. If you use some sheets for notes that you don't wish to be evaluated by the marking team, just put a large cross through the whole sheet, and don't number it.
7. When you've finished, turn in all sheets in proper order (answer sheets first, then used sheets in order, unused sheets and problem text at the bottom) and put them all inside the envelope where you found them; then leave everything on your desk. You are not allowed to take anything out of the room.

## This problem consists of 11 pages (including this one and the answer sheets).

[^9]
## Torsion pendulum

In this experiment we want to study a relatively complex mechanical system - a torsion pendulum - and investigate its main parameters. When its rotation axis is horizontal it displays a simple example of bifurcation.

## Available equipment

1. A torsion pendulum, consisting of an outer body (not longitudinally uniform) and an inner threaded rod, with a stand as shown in figure 1
2. A steel wire with handle
3. A long hexagonal nut that can be screwed onto the pendulum threaded rod (needed only for the last exercise)
4. A ruler and a right triangle template
5. A timer
6. Hexagonal wrenches
7. A3 Millimeter paper sheets.
8. An adjustable clamp
9. Adhesive tape
10. A piece of T-shaped rod

The experimental apparatus is shown in figure 1; it is a torsion pendulum that can oscillate either around a horizontal rotation axis or around a vertical rotation axis. The rotation axis is defined by a short steel wire kept in tension. The pendulum has an inner part that is a threaded rod that may be screwed in and out, and can be fixed in place by means of a small hexagonal lock nut. This threaded rod can not be extracted from the pendulum body.

When assembling the apparatus in step 5 the steel wire must pass through the brass blocks and through the hole in the pendulum, then must be locked in place by keeping it stretched: lock it first at one end, then use the handle to put it in tension and lock it at the other end.

Warning: The wire must be put in tension only to guarantee the pendulum stability. It's not necessary to strain it with a force larger than about 30 N . While straining it, don't bend the wire against the stand, because it might break.


Figure 1: Sketch of the experimental apparatus when its rotation axis is horizontal.
The variables characterizing the pendulum oscillations are:

- the pendulum position defined by the angle $\theta$ of deviation from the direction perpendicular to the plane of the stand frame, which is shown horizontal in figure 1.
- the distance $x$ between the free end of the inner threaded rod and the pendulum rotation axis
- the period $T$ of the pendulum oscillations.

The parameters characterizing the system are:

- the torsional elastic constant $\kappa$ (torque $=\kappa \cdot$ angle) of the steel wire;
- the masses $M_{1}$ and $M_{2}$ of the two parts of the pendulum (1: outer cylinder ${ }^{1}$ and 2 : threaded rod);

[^10]- the distances $R_{1}$ and $R_{2}$ of the center of mass of each pendulum part (1: outer cylinder and 2: threaded rod) from the rotation axis. In this case the inner mobile part (the threaded rod) is sufficiently uniform for computing $R_{2}$ on the basis of its mass, its length $\ell$ and the distance $x . R_{2}$ is therefore a simple function of the other parameters;
- the moments of inertia $I_{1}$ and $I_{2}$ of the two pendulum parts (1: outer cylinder and 2: threaded rod). In this case also we assume that the mobile part (the threaded rod) is sufficiently uniform for computing $I_{2}$ on the basis of its mass, its length $\ell$ and the distance $x$. $I_{2}$ is therefore also a simple function of the other parameters;
- the angular position $\theta_{0}$ (measured between the pendulum and the perpendicular to the plane of the stand frame) where the elastic recall torque is zero. The pendulum is locked to the rotation axis by means of a hex screw, opposite to the threaded rod; therefore $\theta_{0}$ varies with each installation of the apparatus.

Summing up, the system is described by 7 parameters: к, $M_{1}, M_{2}, R_{1}, I_{1}, \ell, \theta_{0}$, but $\theta_{0}$ changes each time the apparatus is assembled, so that only 6 of them are really constants and the purpose of the experiment is that of determining them, namely $\kappa, M_{1}, M_{2}, R_{1}, I_{1}, \ell$, experimentally. Please note that the inner threaded rod can't be drawn out of the pendulum body, and initially only the total mass $M_{1}+M_{2}$ is given (it is printed on each pendulum).

In this experiment several quantities are linear functions of one variable, and you must estimate the parameters of these linear functions. You can use a linear fit, but alternative approaches are also acceptable. The experimental uncertainties of the parameters can be estimated from the procedure of the linear fit or from the spread of experimental data about the fit.

The analysis also requires a simple formula for the moment of inertia of the inner part (we assume that its transverse dimensions are negligible with respect to its length, see figure 2):

$$
\begin{equation*}
I_{2}(x)=\int_{x-\ell}^{x} \lambda s^{2} d s=\frac{\lambda}{3}\left(x^{3}-(x-\ell)^{3}\right)=\frac{\lambda}{3}\left(3 \ell x^{2}-3 \ell^{2} x+\ell^{3}\right) \tag{1}
\end{equation*}
$$

where $\lambda=M_{2} / \ell$ is the linear mass density, and therefore

$$
\begin{equation*}
I_{2}(x)=M_{2} x^{2}-M_{2} \ell x+\frac{M_{2}}{3} \ell^{2} \tag{2}
\end{equation*}
$$



Figure 2: In the analysis of the experiment we can use an equation (eq. 2) for the moment of inertia of a bar whose transverse dimensions are much less than its length. The moment of inertia must be computed about the rotation axis that in this figure crosses the $s$ axis at $s=0$.

Now follow these steps to find the 6 parameters $M_{1}, M_{2}, \kappa, R_{1}, \ell, I_{1}$ :

1. The value of the total mass $M_{1}+M_{2}$ is given (it is printed on the pendulum), and you can find $M_{1}$ and $M_{2}$ by measuring the distance $R(x)$ between the rotation axis and the center of mass of the pendulum. To accomplish this write first an equation for the position $R(x)$ of the center of mass as a function of $x$ and of the parameters $M_{1}, M_{2}, R_{1}, \ell$. points]
2. Now measure $R(x)$ for several values of $x$ (at least 3 ) ${ }^{2}$. Clearly such measurement must be carried out when the pendulum is not attached to the steel wire. Use these measurements and the previous result to find $M_{1}$ and $M_{2}$. [3 points]


Figure 3: The variables $\theta$ and $x$ and the parameters $\theta_{0}$ and $\ell$ are shown here.
3. Find an equation for the pendulum total moment of inertia $I$ as a function of $x$ and of the parameters $M_{2}, I_{1}$ and $\ell$. [0.5 points]
4. Write the pendulum equation of motion in the case of a horizontal rotation axis, as a function of the angle $\theta$ (see figure 3) and of $x, \kappa, \theta_{0}, M_{1}, M_{2}$, the total moment of inertia $I$ and the position $R(x)$ of the center of mass. [1 point]

[^11]5. In order to determine $\kappa$, assemble now the pendulum and set it with its rotation axis horizontal. The threaded rod must initially be as far as possible inside the pendulum. Lock the pendulum to the steel wire, with the hex screw, at about half way between the wire clamps and in such a way that its equilibrium angle (under the combined action of weight and elastic recall) deviates sizeably from the vertical (see figure 4). Measure the equilibrium angle $\theta_{e}$ for several values of $x$ (at least 5). [4 points]


Figure 4: In this measurement set the pendulum so that its equilibrium position deviates from the vertical.
6. Using the last measurements, find $\kappa$. [4.5 points]
7. Now place the pendulum with its rotation axis vertical ${ }^{3}$, and measure its oscillation period for several values of $x$ (at least 5). With these measurements, find $I_{1}$ and $\ell$. [4 points]

At this stage, after having found the system parameters, set the experimental apparatus as follows:

- pendulum rotation axis horizontal
- threaded rod as far as possible inside the pendulum
- pendulum as vertical as possible near equilibrium
- finally add the long hexagonal nut at the end of the threaded rod by screwing it a few turns (it can't go further than that)

In this way the pendulum may have two equilibrium positions, and the situation varies according to the position of the threaded rod, as you can also see from the generic graph shown in figure 5, of the potential energy as a function of the angle $\theta$.

The doubling of the potential energy minimum in figure 5 illustrates a phenomenon known in mathematics as bifurcation; it is also related to the various kinds of symmetry breaking that are studied in particle physics and statistical mechanics.

[^12]

Figure 5: Graph of the function $U(\theta)=\frac{a}{2}\left(\theta-\theta_{0}\right)^{2}+\cos \theta$ (which is proportional to the potential energy of this problem) as a function of $\theta$, with $\theta_{0} \neq 0$. The various curves correspond to different $a$ values as labeled in the figure; smaller values of $a \quad(a<1)$ correspond to the appearence of the bifurcation. In our case the parameter $a$ is associated with the position $x$ of the threaded rod.

We can now study this bifurcation by measuring the period of the small oscillations about the equilibrium position:
8. Plot the period ${ }^{4} T$ as a function of $x$. What kind of function is it? Is it increasing, decreasing or is it a more complex function? [2.5 points]

[^13]
## Solution

1. At equilibrium the pressure $p$ inside the vessel must be equal to the room pressure $p_{0}$ plus the pressure induced by the weight of the movable base: $p=p_{0}+\frac{m g}{\pi r^{2}}$. This is true before and after irradiation. Initially the gas temperature is room temperature. Owing to the state equation of perfect gases, the initial gas volume $V_{1}$ is $V_{1}=\frac{n R T_{0}}{p}$ (where $R$ is the gas constant) and therefore the height $h_{1}$ of the cylinder which is occupied by the gas is $h_{1}=\frac{V_{1}}{\pi r^{2}}=\frac{n R T_{0}}{p_{0} \pi r^{2}+m g}$. After irradiation, this height becomes $h_{2}=h_{1}+\Delta s$, and therefore the new temperature is $T_{2}=T_{0}\left(1+\frac{\Delta s}{h_{1}}\right)=T_{0}+\frac{\Delta s\left(p_{0} \pi r^{2}+m g\right)}{n R}$.
Numerical values: $p=102.32 \mathrm{kPa} ; T_{2}=322 \mathrm{~K}=49^{\circ} \mathrm{C}$
2. The mechanical work made by the gas against the plate weight is $m g \Delta s$ and against the room pressure is $p_{0} \pi r^{2} \Delta s$, therefore the total work is $L=\left(m g+p_{0} \pi r^{2}\right) \Delta s=24.1 \mathrm{~J}$
3. The internal energy, owing to the temperature variation, varies by an amount $\Delta U=n c_{\mathrm{V}}\left(T_{2}-T_{0}\right)$.

The heat introduced into the system during the irradiation time $\Delta t$ is $Q=\Delta U+L=n c_{\mathrm{V}} \frac{T_{0} \Delta s}{h_{1}}+\left(m g+p_{0} \pi r^{2}\right) \Delta s=\Delta s\left(p_{0} \pi r^{2}+m g\right)\left(\frac{c_{\mathrm{V}}}{R}+1\right)$. This heat comes exclusively from the absorption of optical radiation and coincides therefore with the absorbed optical energy, $Q=84 \mathrm{~J}$.

The same result can also be obtained by considering an isobaric transformation and remembering the relationship between molecular heats:
$Q=n c_{p}\left(T_{2}-T_{0}\right)=n\left(c_{\mathrm{V}}+R\right)\left[\frac{\Delta s\left(p_{0} \pi r^{2}+m g\right)}{n R}\right]=\Delta s\left(p_{0} \pi r^{2}+m g\right)\left(\frac{c_{\mathrm{V}}}{R}+1\right)$
4. Since the laser emits a constant power, the absorbed optical power is $W=\frac{Q}{\Delta t}=\left(\frac{c_{\mathrm{V}}}{R}+1\right) \frac{\Delta s}{\Delta t}\left(p_{0} \pi r^{2}+m g\right)=8.4 \mathrm{~W}$. The energy of each photon is $h c / \lambda$, and thus the number of photons absorbed per unit time is $\frac{W \lambda}{h c}=2.2 \cdot 10^{19} \mathrm{~s}^{-1}$
5. The potential energy change is equal to the mechanical work made against the plate weight, therefore the efficiency $\eta$ of the energy transformation is

Problem 1 - Solution

$$
\frac{m g \Delta s}{Q}=\frac{1}{\left(1+\frac{p_{0} \pi r^{2}}{m g}\right)\left(1+\frac{c_{V}}{R}\right)}=2.8 \cdot 10^{-3} \approx 0.3 \%
$$

6. When the cylinder is rotated and its axis becomes horizontal, we have an adiabatic transformation where the pressure changes from $p$ to $p_{0}$, and the temperature changes therefore to a new value $T_{3}$. The equation of the adiabatic transformation $p V^{\gamma}=$ constant may now be written in the form $T_{3}=T_{2}\left(\frac{p_{0}}{p}\right)^{\frac{\gamma-1}{\gamma}}$, where $\gamma=\frac{c_{\mathrm{p}}}{c_{\mathrm{V}}}=\frac{c_{\mathrm{V}}+R}{c_{\mathrm{V}}}=1+\frac{R}{c_{\mathrm{V}}}=1.399$. Finally $T_{3}=321 \mathrm{~K}=48^{\circ} \mathrm{C}$

## Grading guidelines

1. 0.5 Understanding the relationship between inner and outer pressure
0.7 Proper use of the plate displacement
$0.2+0.2$ Correct results for final pressure
$0.2+0.2$ Correct results for final temperature
2. 0.6 Understanding that the work is made both against plate weight and against atmospheric pressure
0.2+0.2 Correct results for work
3. 1 Correct approach
$0.5 \quad$ Correct equation for heat
0.3 Understanding that the absorbed optical energy equals heat
0.2 Correct numerical result for optical energy
4. $0.2+0.2$ Correct results for optical power
$0.5 \quad$ Einstein's equation
$0.3+0.3$ Correct results for number of photons
5. 0.6 Computation of the change in potential energy
$0.2+0.2$ Correct results for efficiency
6. $0.8 \quad$ Understanding that the pressure returns to room value
0.4 Understanding that there is an adiabatic transformation
$0.4 \quad$ Equation of adiabatic transformation
0.5 Derivation of $\gamma$ from the relationship between specific heats
0.2+0.2 Correct results for temperature

For "correct results" two possible marks are given: the first one is for the analytical equation and the second one for the numerical value.
For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing. No bonus can be given for taking into account the gas weight

## Solution

1. The contribution to $\mathbf{B}$ given by each leg of the " V " has the same direction as that of a corresponding infinite wire and therefore - if the current proceeds as indicated by the arrow - the magnetic field is orthogonal to the wire plane taken as the $x-y$ plane. If we use a right-handed reference frame as indicated in the figure, $\mathbf{B}(\mathrm{P})$ is along the positive $z$ axis.


For symmetry reasons, the total field is twice that generated by each leg and has still the same direction.

2A. When $\alpha=\pi / 2$ the " V " becomes a straight infinite wire. In this case the magnitude of the field $B(\mathrm{P})$ is known to be $B=\frac{i}{2 \pi \varepsilon_{0} c^{2} d}=\frac{i \mu_{0}}{2 \pi d}$, and since $\tan (\pi / 4)=1$, the factor $k$ is $\frac{i \mu_{0}}{2 \pi d}$.

## The following solution is equally acceptable:

2B. If the student is aware of the equation $B=\frac{\mu_{0} i}{4 \pi} \frac{\cos \theta_{1}-\cos \theta_{2}}{h}$ for a finite stretch of wire lying on a straight line at a distance $h$ from point $P$ and whose ends are seen from $P$ under the angles $\theta_{1}$ and $\theta_{2}$, he can find that the two legs of the " V " both produce fields $\frac{\mu_{0} i}{4 \pi} \frac{1-\cos \alpha}{d \sin \alpha}$ and therefore the total field is $B=\frac{i \mu_{0}}{2 \pi d} \frac{1-\cos \alpha}{\sin \alpha}=\frac{i \mu_{0}}{2 \pi d} \tan \left(\frac{\alpha}{2}\right)$. This is a more complete solution since it also proves the angular dependence, but it is not required.

3A. In order to compute $\mathbf{B}\left(\mathrm{P}^{*}\right)$ we may consider the " V " as equivalent to two crossed infinite wires ( $a$ and $b$ in the following figure) plus another " V ", symmetrical to the first one, shown in the figure as $\mathrm{V}^{\prime}$, carrying the same current $i$, in opposite direction.


Then $B\left(\mathrm{P}^{*}\right)=B_{a}\left(\mathrm{P}^{*}\right)+B_{b}\left(\mathrm{P}^{*}\right)+B_{\mathrm{V}^{\prime}}\left(\mathrm{P}^{*}\right)$. The individual contributions are:
$B_{a}\left(\mathrm{P}^{*}\right)=B_{b}\left(\mathrm{P}^{*}\right)=\frac{i \mu_{0}}{2 \pi d \sin \alpha}$, along the negative $z$ axis;
$B_{\mathrm{V}}\left(\mathrm{P}^{*}\right)=\frac{i \mu_{0}}{2 \pi d} \tan \left(\frac{\alpha}{2}\right)$, along the positive $z$ axis.
Therefore we have $B\left(\mathrm{P}^{*}\right)=\frac{i \mu_{0}}{2 \pi d}\left[\frac{2}{\sin \alpha}-\tan \left(\frac{\alpha}{2}\right)\right]=k\left(\frac{1+\cos \alpha}{\sin \alpha}\right)=k \cot \left(\frac{\alpha}{2}\right)$, and the field is along the negative $z$ axis.

The following solutions are equally acceptable:
3B. The point $\mathrm{P}^{*}$ inside a "V" with half-span $\alpha$ can be treated as if it would be on the outside of a "V" with half-span $\pi-\alpha$ carrying the same current but in an opposite way, therefore the field is $B\left(\mathrm{P}^{*}\right)=k \tan \left(\frac{\pi-\alpha}{2}\right)=k \tan \left(\frac{\pi}{2}-\frac{\alpha}{2}\right)=k \cot \left(\frac{\alpha}{2}\right)$; the direction is still that of the $z$ axis but it is along the negative axis because the current flows in the opposite way as previously discussed.

3C. If the student follows the procedure outlined under 2B., he/she may also find the field value in $\mathrm{P}^{*}$ by the same method.
4. The mechanical moment $\mathbf{M}$ acting on the magnetic needle placed in point $P$ is given by $\mathbf{M}=\boldsymbol{\mu} \wedge \mathbf{B}$ (where the symbol $\wedge$ is used for vector product). If the needle is displaced from its equilibrium position by an angle $\beta$ small enough to approximate $\sin \beta$ with $\beta$, the angular momentum theorem gives $M=-\mu \mathrm{B} \beta=\frac{d L}{d t}=I \frac{d^{2} \beta}{d t^{2}}$, where there is a minus sign because the mechanical momentum is always opposite to the displacement from equilibrium. The period $T$ of the small oscillations is therefore given by $T=\frac{2 \pi}{\omega}=2 \pi \sqrt{\frac{I}{\mu B}}$.
Writing the differential equation, however, is not required: the student should recognise the same situation as with a harmonic oscillator.
5. If we label with subscript A the computations based on Ampère's interpretation, and with subscript BS those based on the other hypothesis by Biot and Savart, we have
$B_{\mathrm{A}}=\frac{i \mu_{0}}{2 \pi d} \tan \left(\frac{\alpha}{2}\right)$
$B_{\mathrm{BS}}=\frac{i \mu_{0}}{\pi^{2} d} \alpha$
$T_{\mathrm{A}}=2 \pi \sqrt{\frac{2 \pi I d}{\mu_{0} \mu i \tan \left(\frac{\alpha}{2}\right)}}$
$T_{\mathrm{BS}}=2 \pi \sqrt{\frac{\pi^{2} I d}{\mu_{0} \mu i \alpha}}$
$\frac{T_{\mathrm{A}}}{T_{\mathrm{BS}}}=\sqrt{\frac{2 \alpha}{\pi \tan \left(\frac{\alpha}{2}\right)}}$

For $\alpha=\pi / 2$ (maximum possible value) $T_{\mathrm{A}}=T_{\mathrm{BS}}$; and for $\alpha \rightarrow 0 T_{\mathrm{A}} \rightarrow \frac{2}{\sqrt{\pi}} T_{\mathrm{BS}} \approx 1.128 T_{\mathrm{BS}}$. Since within this range $\frac{\tan (\alpha / 2)}{\alpha / 2}$ is a monotonically growing function of $\alpha, \frac{T_{\mathrm{A}}}{T_{\mathrm{BS}}}$ is a monotonically decreasing function of $\alpha$; in an experiment it is therefore not possible to distinguish between the two interpretations if the value of $\alpha$ is larger than the value for which $T_{\mathrm{A}}=1.10 T_{\mathrm{BS}}(10 \%$ difference), namely when $\tan \left(\frac{\alpha}{2}\right)=\frac{4}{1.21 \pi} \frac{\alpha}{2}=1.05 \frac{\alpha}{2}$. By looking into the trigonometry tables or using a calculator we see that this condition is well approximated when $\alpha / 2=0.38 \mathrm{rad}$; $\alpha$ must therefore be smaller than $0.77 \mathrm{rad} \approx 44^{\circ}$.
A graphical solution of the equation for $\alpha$ is acceptable but somewhat lengthy. A series development, on the contrary, is not acceptable.

## Grading guidelines

1. 1 for recognising that each leg gives the same contribution
0.5 for a correct sketch
2. 0.5 for recognising that $\alpha=\pi / 2$ for a straight wire, or for knowledge of the equation given in 2B.
0.25 for correct field equation (infinite or finite)
0.25 for value of $k$
3. 0.7 for recognising that the V is equivalent to two infinite wires plus another V
0.3 for correct field equation for an infinite wire
0.5 for correct result for the intensity of the required field
0.5 for correct field direction
alternatively
0.8 for describing the point as outside a V with $\pi$ - $\alpha$ half-amplitude and opposite current
0.7 for correct analytic result
0.5 for correct field direction
alternatively
0.5 for correctly using equation under 2B

1 for correct analytic result
0.5 for correct field direction
4. 0.5 for correct equation for mechanical moment $\mathbf{M}$
0.5 for doing small angle approximation $\sin \beta \approx \beta$

1 for correct equation of motion, including sign, or for recognizing analogy with harmonic oscillator
0.5 for correct analytic result for $T$
5. 0.3 for correct formulas of the two periods
0.3 for recognising the limiting values for $\alpha$
0.4 for correct ratio between the periods

1 for finding the relationship between $\alpha$ and tangent
0.5 for suitable approximate value of $\alpha$
0.5 for final explicit limiting value of $\alpha$

For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing

## Solution

1A. Assuming - as outlined in the text - that the orbit is circular, and relating the radial acceleration $\frac{V^{2}}{R}$ to the gravitational field $\frac{G M_{\mathrm{S}}}{R^{2}}$ (where $M_{\mathrm{S}}$ is the solar mass) we obtain Jupiter's orbital speed $V=\sqrt{\frac{G M_{\mathrm{S}}}{R}} \approx 1.306 \cdot 10^{4} \mathrm{~m} / \mathrm{s}$.

The following alternative solution is also acceptable:
1B. Since we treat Jupiter's motion as circular and uniform, $V=\omega R=\frac{2 \pi R}{y_{\mathrm{J}}}$, where $y_{\mathrm{J}}$ is the revolution period of Jupiter, which is given in the list of the general physical constants.
2. The two gravitational forces on the space probe are equal when

$$
\begin{equation*}
\frac{G M m}{\rho^{2}}=\frac{G M_{\mathrm{S}} m}{(R-\rho)^{2}} \tag{2}
\end{equation*}
$$

(where $\rho$ is the distance from Jupiter and $M$ is Jupiter's mass), whence

$$
\begin{equation*}
\sqrt{M}(R-\rho)=\rho \sqrt{M_{\mathrm{S}}} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\rho=\frac{\sqrt{M}}{\sqrt{M_{\mathrm{S}}}+\sqrt{M}} R=0.02997 R=2.333 \cdot 10^{10} \mathrm{~m} \tag{4}
\end{equation*}
$$

and therefore the two gravitational attractions are equal at a distance of about 23.3 million kilometers from Jupiter (about 334 Jupiter radii).
3. With a simple Galilean transformation we find that the velocity components of the probe in Jupiter's reference frame are

$$
\left\{\begin{array}{l}
v_{x}^{\prime}=V \\
v_{y}^{\prime}=v_{0}
\end{array}\right.
$$

and therefore - in Jupiter's reference frame - the probe travels with an angle $\theta_{0}=\arctan \frac{v_{0}}{V}$ with respect to the $x$ axis and its speed is $v^{\prime}=\sqrt{v_{0}^{2}+V^{2}}$ (we also note that $\cos \theta_{0}=\frac{V}{\sqrt{v_{0}^{2}+V^{2}}}=\frac{V}{v^{\prime}}$
and $\sin \theta_{0}=\frac{v_{0}}{\sqrt{v_{0}^{2}+V^{2}}}=\frac{v_{0}}{v^{\prime}}$.
Using the given values we obtain $\theta_{0}=0.653 \mathrm{rad} \approx 37.4^{\circ}$ and $v^{\prime}=1.65 \cdot 10^{4} \mathrm{~m} / \mathrm{s}$.
4. Since the probe trajectory can be described only approximately as the result of a two-body gravitational interaction (we should also take into account the interaction with the Sun and other planets) we assume a large but not infinite distance from Jupiter and we approximate the total energy in Jupiter's reference frame as the probe's kinetic energy at that distance:

$$
\begin{equation*}
E \approx \frac{1}{2} m v^{\prime 2} \tag{5}
\end{equation*}
$$

The corresponding numerical value is $E=112 \mathrm{GJ}$.
5. Equation (1) shows that the radial distance becomes infinite, and its reciprocal equals zero, when

$$
\begin{equation*}
1+\sqrt{1+\frac{2 E v^{\prime 2} b^{2}}{G^{2} M^{2} m}} \cos \theta=0 \tag{7}
\end{equation*}
$$

namely when

$$
\begin{equation*}
\cos \theta=-\frac{1}{\sqrt{1+\frac{2 E v^{\prime 2} b^{2}}{G^{2} M^{2} m}}} \tag{8}
\end{equation*}
$$

We should also note that the radial distance can't be negative, and therefore its acceptable values are those satisfying the equation

$$
\begin{equation*}
1+\sqrt{1+\frac{2 E v^{\prime 2} b^{2}}{G^{2} M^{2} m}} \cos \theta \geq 0 \tag{9}
\end{equation*}
$$

or

$$
\begin{equation*}
\cos \theta \geq-\frac{1}{\sqrt{1+\frac{2 E v^{\prime 2} b^{2}}{G^{2} M^{2} m}}} \tag{10}
\end{equation*}
$$

The solutions for the limiting case of eq. (10) (i.e. when the equal sign applies) are:

$$
\begin{equation*}
\theta_{ \pm}= \pm \arccos \left[-\left(1+\frac{2 E v^{\prime 2} b^{2}}{G^{2} M^{2} m}\right)^{-1 / 2}\right]= \pm\left(\pi-\arccos \frac{1}{\sqrt{1+\frac{2 E v^{\prime 2} b^{2}}{G^{2} M^{2} m}}}\right) \tag{11}
\end{equation*}
$$

and therefore the angle $\Delta \theta$ (shown in figure 2 ) between the two hyperbola asymptotes is given by:

$$
\begin{align*}
\Delta \theta & =\left(\theta_{+}-\theta_{-}\right)-\pi \\
& =\pi-2 \arccos \frac{1}{\sqrt{1+\frac{2 E v^{\prime 2} b^{2}}{G^{2} M^{2} m}}}  \tag{12}\\
& =\pi-2 \arccos \frac{1}{\sqrt{1+\frac{v^{\prime 4} b^{2}}{G^{2} M^{2}}}}
\end{align*}
$$

In the last line, we used the value of the total energy as computed in the previous section.
6. The angular deviation is a monotonically decreasing function of the impact parameter, whence the deviation has a maximum when the impact parameter has a minimum. From the discussion in the previous section we easily see that the point of nearest approach is when $\theta=0$, and in this case the minimum distance between probe and planet center is easily obtained from eq. (1):

$$
\begin{equation*}
r_{\min }=\frac{v^{\prime 2} b^{2}}{G M}\left(1+\sqrt{1+\frac{v^{\prime 4} b^{2}}{G^{2} M^{2}}}\right)^{-1} \tag{13}
\end{equation*}
$$

By inverting equation (13) we obtain the impact parameter

$$
\begin{equation*}
b=\sqrt{r_{\min }^{2}+\frac{2 G M}{v^{\prime 2}} r_{\text {min }}} \tag{14}
\end{equation*}
$$

We may note that this result can alternatively be obtained by considering that, due to the conservation of angular momentum, we have

$$
L=m v^{\prime} b=m v_{\min }^{\prime} r_{\min }
$$

where we introduced the speed corresponding to the nearest approach. In addition, the conservation of energy gives

$$
E=\frac{1}{2} m v^{\prime 2}=\frac{1}{2} m v_{\min }^{\prime 2}-\frac{G M m}{r_{\min }}
$$

and by combining these two equations we obtain equation (14) again.
The impact parameter is an increasing function of the distance of nearest approach; therefore, if the probe cannot approach Jupiter's surface by less than two radii (and thus $r_{\min }=$ $3 R_{\mathrm{B}}$, where $R_{\mathrm{B}}$ is Jupiter's body radius), the minimum acceptable value of the impact parameter is

$$
\begin{equation*}
b_{\min }=\sqrt{9 R_{\mathrm{B}}^{2}+\frac{6 G M}{v^{\prime 2}} R_{\mathrm{B}}} \tag{15}
\end{equation*}
$$

From this equation we finally obtain the maximum possible deviation:

$$
\begin{equation*}
\Delta \theta_{\max }=\pi-2 \arccos \frac{1}{\sqrt{1+\frac{v^{\prime 4} b_{\min }^{2}}{G^{2} M^{2}}}}=\pi-2 \arccos \frac{1}{\sqrt{1+\frac{v^{\prime 4}}{G^{2} M^{2}}\left(9 R_{\mathrm{B}}^{2}+\frac{6 G M}{v^{\prime 2}} R_{\mathrm{B}}\right)}} \tag{16}
\end{equation*}
$$

and by using the numerical values we computed before we obtain:
$b_{\text {min }}=4.90 \cdot 10^{8} \mathrm{~m} \approx 7.0 R_{\mathrm{B}}$ and $\quad \Delta \theta_{\text {max }}=1.526 \mathrm{rad} \approx 87.4^{\circ}$
7. The final direction of motion with respect to the $x$ axis in Jupiter's reference frame is given by the initial angle plus the deviation angle, thus $\boldsymbol{\omega}+\Delta \boldsymbol{\theta}$ if the probe passes behind the planet. The final velocity components in Jupiter's reference frame are therefore:

$$
\left\{\begin{array}{l}
v_{x}^{\prime}=v^{\prime} \cos \left(\theta_{0}+\Delta \theta\right) \\
v_{y}^{\prime}=v^{\prime} \sin \left(\theta_{0}+\Delta \theta\right)
\end{array}\right.
$$

whereas in the Sun reference frame they are

$$
\left\{\begin{array}{c}
v_{x}^{\prime \prime}=v^{\prime} \cos \left(\theta_{0}+\Delta \theta\right)-V \\
v_{y}^{\prime \prime}=v^{\prime} \sin \left(\theta_{0}+\Delta \theta\right)
\end{array}\right.
$$

Therefore the final probe speed in the Sun reference frame is

$$
\begin{align*}
v^{\prime \prime} & =\sqrt{\left(v^{\prime} \cos \left(\theta_{0}+\Delta \theta\right)-V\right)^{2}+\left(v^{\prime} \sin \left(\theta_{0}+\Delta \theta\right)\right)^{2}} \\
& =\sqrt{v_{0}^{2}+2 V^{2}-2 v^{\prime} V \cos \left(\theta_{0}+\Delta \theta\right)} \\
& =\sqrt{v_{0}^{2}+2 V^{2}-2 v^{\prime} V\left(\cos \theta_{0} \cos \Delta \theta-\sin \theta_{0} \sin \Delta \theta\right)}  \tag{17}\\
& =\sqrt{v_{0}^{2}+2 V^{2}-2 V\left(V \cos \Delta \theta-v_{0} \sin \Delta \theta\right)} \\
& =\sqrt{v_{0}\left(v_{0}+2 V \sin \Delta \theta\right)+2 V^{2}(1-\cos \Delta \theta)}
\end{align*}
$$

8. Using the value of the maximum possible angular deviation, the numerical result is $v$ " $=2.62 \cdot 10^{4}$ $\mathrm{m} / \mathrm{s}$.

## Grading guidelines

1. 0.4 Law of gravitation, or law of circular uniform motion
0.4 Correct approach
0.4+0.3 Correct results for velocity of Jupiter
2. 0.3 Correct approach
0.4+0.3 Correct results for distance from Jupiter
3. 1 Correct transformation between reference frames
0.3+0.2 Correct results for probe speed in Jupiter reference frame
0.3+0.2 Correct results for probe angle
4. 0.8 Understanding how to handle the potential energy at infinity
0.2 Numerical result for kinetic energy
5. 0.6 Correct approach
0.6 Equation for the orientation of the asymptotes
0.8 Equation for the probe deflection angle
6. $0.3+0.2$ Correct results for minimum impact parameter
0.3+0.2 Correct results for maximum deflection angle
7. 0.5 Equation for velocity components in the Sun reference frame
0.5 Equation for speed as a function of angular deflection
8. $\quad 0.5$ Numerical result for final speed

For "correct results" two possible marks are given: the first one is for the analytical equation and the second one for the numerical value.
For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing.

## Solution

The numerical values given in the text are those obtained in a preliminary test performed by a student of the University of Bologna ${ }^{1}$, and are reported here only as a guide to the evaluation of the student solutions.

1. and 2. The distance from the center of mass to the rotation axis is:
$R(x)=\frac{M_{1} R_{1}+M_{2}(x-\ell / 2)}{M_{1}+M_{2}}$
and therefore, if we measure the position of the center of mass ${ }^{2}$ as a function of $x$ we obtain a relationship between the system parameters, and by a linear fit of eq. (1) we obtain an angular coefficient equal to $M_{2} /\left(M_{1}+M_{2}\right)$, and from these equations, making use of the given total mass $M_{1}+M_{2}=41.0 \mathrm{~g} \pm 0.1 \mathrm{~g}$, we obtain $M_{1}$ and $M_{2}$. The following table shows some results obtained in the test run.

| $n$ | $x[\mathrm{~mm}]$ | $R(x)[\mathrm{mm}]$ |
| :---: | :---: | :---: |
| 1 | $204 \pm 1$ | $76 \pm 1$ |
| 2 | $220 \pm 1$ | $83 \pm 1$ |
| 3 | $236 \pm 1$ | $89 \pm 1$ |
| 4 | $254 \pm 1$ | $95 \pm 1$ |
| 5 | $269 \pm 1$ | $101 \pm 1$ |
| 6 | $287 \pm 1$ | $107 \pm 1$ |
| 7 | $302 \pm 1$ | $113 \pm 1$ |
| 8 | $321 \pm 1$ | $119 \pm 1$ |

Figure 6 shows the data concerning the position of the pendulum's center of mass together with a best fit straight line: the estimated error on the length measurements is now 1 mm and we treat it as a Gaussian error. Notice that both the dependent variable $R(x)$ and the independent variable $x$ are affected by the experimental uncertainty, however we decide to neglect the uncertainty on $x$, since it is smaller than $1 \%$. The coefficients $a$ and $b$ in $R(x)=$ $a x+b$ are
$a=0.366 \pm 0.009$
$b=2 \mathrm{~mm} \pm 2 \mathrm{~mm}$

[^14](therefore $b$ is compatible with 0 )


Figure 6: Graph of the position of the pendulum's center of mass (with respect to the rotation axis) as a function of the variable $x$. The numbering of the data points corresponds to that mentioned in the main text. The estimated error is compatible with the fluctuations of the measured data.

For computing the masses only the $a$ value is needed; using the total pendulum mass we find:
$M_{1}=26.1 \pm 0.4 \mathrm{~g}$
$M_{2}=15.0 \mathrm{~g} \pm 0.4 \mathrm{~g}$

Even though many non-programmable pocket calculators can carry out a linear regression, it is likely that many students will be unable to do such an analysis, and in particular they may be unable to estimate the uncertainty of the fit parameters even if their pocket calculators provide a linear regression mode. It is also acceptable to find $a$ and $b$ using several pairs of measurements and finally computing a weighted average of the results. For each pair of measurements $a$ and $b$ are given by
$a=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}$
$b=y_{2}-a x_{2}$
and the parameter uncertainties (assuming them gaussian) by
$\Delta a=a \sqrt{\frac{\Delta x_{1}^{2}+\Delta x_{2}^{2}}{\left(x_{1}-x_{2}\right)^{2}}+\frac{\Delta y_{1}^{2}+\Delta y_{2}^{2}}{\left(y_{1}-y_{2}\right)^{2}}}$
$\Delta b=\sqrt{\Delta y_{2}^{2}+a^{2} x_{2}^{2}\left(\frac{\Delta x_{2}^{2}}{x_{2}^{2}}+\frac{\Delta a^{2}}{a^{2}}\right)}$

In order to calculate (2) and (3) the data can be paired with a scheme like $\{1,5\},\{2,6\},\{3,7\},\{4,8\}$, where "far" points are coupled in order to minimize the error on each pair.
There may be other alternative and equally acceptable approaches: they should all be considered valid if the order of magnitude of the estimated uncertainty is correct.
3. The pendulum's total moment of inertia is the sum of the moments of its two parts, and from figure 3 we see that

$$
\begin{equation*}
I(x)=I_{1}+I_{2}(x)=M_{2} x^{2}-M_{2} \ell x+\left(I_{1}+\frac{M_{2}}{3} \ell^{2}\right) \tag{4}
\end{equation*}
$$

4. The pendulum's equation of motion is

$$
\begin{equation*}
I(x) \frac{d^{2} \theta}{d t^{2}}=-\kappa\left(\theta-\theta_{0}\right) \tag{5}
\end{equation*}
$$

if the rotation axis is vertical, while it's
$I(x) \frac{d^{2} \theta}{d t^{2}}=-\kappa\left(\theta-\theta_{0}\right)+\left(M_{1}+M_{2}\right) g R(x) \sin \theta$
if the rotation axis is horizontal.
5. and 6. When the system is at rest in an equilibrium position, the angular acceleration is zero and therefore the equilibrium positions $\theta_{e}$ can be found by solving the equation
$-\kappa\left(\theta_{\mathrm{e}}-\theta_{0}\right)+\left(M_{1}+M_{2}\right) g R(x) \sin \theta_{\mathrm{e}}=0$

If the value $x_{i}$ corresponds to the equilibrium angle $\theta_{e, i}$, and if we define the quantity (that can be computed from the experimental data) $y_{i}=\left(M_{1}+M_{2}\right) g R\left(x_{i}\right) \sin \theta_{e, i}$, then eq. (7) may be written as
$y_{i}=\kappa \theta_{e, i}-\kappa \theta_{0}$
and therefore the quantities $\kappa$ and $\kappa \theta_{0}$ can be found with a linear fit. The following table shows several data collected in a trial run according to the geometry shown in figure 7.

| $n$ | $x$ [mm] | $h$ [mm] | $\sin \theta_{\mathrm{e}}=h / x$ | $\theta_{\text {e }}$ | $y[\mathrm{~N} \cdot \mu \mathrm{~m}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $204 \pm 1$ | $40 \pm 1$ | $0.196 \pm 0.005$ | $0.197 \pm 0.005$ | $6.1 \pm 0.3$ |
| 2 | $220 \pm 1$ | $62 \pm 1$ | $0.282 \pm 0.005$ | $0.286 \pm 0.005$ | $9.4 \pm 0.4$ |
| 3 | $238 \pm 1$ | $75 \pm 1$ | $0.315 \pm 0.004$ | $0.321 \pm 0.005$ | $11.3 \pm 0.5$ |
| 4 | 255 $\pm 1$ | $89 \pm 1$ | $0.349 \pm 0.004$ | $0.357 \pm 0.004$ | $13.4 \pm 0.5$ |
| 5 | $270 \pm 1$ | $109 \pm 1$ | $0.404 \pm 0.004$ | $0.416 \pm 0.004$ | $16.4 \pm 0.6$ |
| 6 | $286 \pm 1$ | $131 \pm 1$ | $0.458 \pm 0.004$ | $0.476 \pm 0.004$ | $19.7 \pm 0.7$ |
| 7 | $307 \pm 1$ | $162 \pm 1$ | $0.528 \pm 0.004$ | $0.556 \pm 0.004$ | $24.3 \pm 0.8$ |
| 8 | $321 \pm 1$ | $188 \pm 1$ | $0.586 \pm 0.004$ | $0.626 \pm 0.004$ | $28.2 \pm 0.9$ |



Figure 7: Geometry of the measurements taken for finding the angle.

We see that not only the dependent but also the independent variable is affected by a measurement uncertainty, but the relative uncertainty on $\theta_{\mathrm{e}}$ is much smaller than the relative uncertainty on $y$ and we neglect it. We obtain from such data (neglecting the first data point, see figure 8):
$\kappa=0.055 \mathrm{~N} \cdot \mathrm{~m} \cdot \mathrm{rad}^{-1} \pm 0.001 \mathrm{~N} \cdot \mathrm{~m} \cdot \mathrm{rad}^{-1}$
$\kappa \theta_{0}=-0.0063 \mathrm{~N} \cdot \mathrm{~m} \pm 0.0008 \mathrm{~N} \cdot \mathrm{~m}$

Clearly in this case only the determination of the torsion coefficient $\kappa$ is interesting. The fit of the experimental data is shown in figure 8.


Figure 8: Fit of eq. (8) as a function of $\theta$. In this case the estimated error is again compatible with the experimental data fluctuations. However the data points show a visible deviation from straightness which may be due to an error in the first measurement (the one at lowest $\theta$ ).
7. The moment of inertia can be found experimentally using the pendulum with its rotation axis vertical and recalling eq. (5); from this equation we see that the pendulum oscillates with angular frequency $\omega(x)=\sqrt{\frac{\kappa}{I(x)}}$ and therefore

$$
\begin{equation*}
I(x)=\frac{\kappa T^{2}(x)}{4 \pi^{2}} \tag{9}
\end{equation*}
$$

where $T$ is the measured oscillation period. Using eq. (9) we see that eq. (4) can be rewritten as

$$
\begin{equation*}
\frac{\kappa}{4 \pi^{2}} T^{2}(x)-M_{2} x^{2}=-M_{2} \ell x+\left(I_{1}+\frac{M_{2}}{3} \ell^{2}\right) \tag{10}
\end{equation*}
$$

The left-hand side in eq. (10) is known experimentally, and therefore with a simple linear fit we can find the coefficients $M_{2} \ell$ and $\left(I_{1}+\frac{M_{2}}{3} \ell^{2}\right)$, as we did before. The experimental data are in this case:

| $n$ | $x[\mathrm{~mm}]$ | $T[\mathrm{~s}]$ |
| :---: | :---: | :---: |
| 1 | $204 \pm 1$ | $0.502 \pm 0.002$ |
| 2 | $215 \pm 1$ | $0.528 \pm 0.002$ |
| 3 | $231 \pm 1$ | $0.562 \pm 0.002$ |
| 4 | $258 \pm 1$ | $0.628 \pm 0.002$ |
| 5 | $290 \pm 1$ | $0.708 \pm 0.002$ |
| 6 | $321 \pm 1$ | $0.790 \pm 0.002$ |

The low uncertainty on $T$ has been obtained measuring the total time required for 50 full periods.
Using the previous data and another linear fit, we find
$\ell=230 \mathrm{~mm} \pm 20 \mathrm{~mm}$
$I_{1}=1.7 \cdot 10^{-4} \mathrm{~kg} \cdot \mathrm{~m}^{2} \pm 0.7 \cdot 10^{-4} \mathrm{~kg} \cdot \mathrm{~m}^{2}$
and the fit of the experimental data is shown in figure 9.

$$
y=\frac{\kappa}{4 \pi^{2}} T^{2}(x)-M_{2} x^{2}\left[\mathrm{~kg} \cdot \mathrm{~m}^{2}\right] \text { vs. } x[\mathrm{~m}]
$$



Figure 9: Fit of eq. (10) as a function of $x$. In this case the estimated error is again compatible with the experimental data fluctuations.
8. Although in this case the period $T$ is a complicated function of $x$, its graph is simple, and it is shown in figure 10 , along with the test experimental data.

The required answer is that there is a single local maximum.


Figure 10: The period $T$ of the pendulum with horizontal axis as a function of $x$. In addition to the experimental points the figure shows the result of a theoretical calculation of the period in which the following values have been assumed: $g=9.81 \mathrm{~m} / \mathrm{s}^{2} ; \kappa=0.056 \mathrm{~N} \cdot \mathrm{~m} / \mathrm{rad} ; M_{1}=0.0261 \mathrm{~kg} ; M_{2}=$ $0.0150 \mathrm{~kg} ; M_{3}=0.00664 \mathrm{~kg} ; I_{1}=1.0 \cdot 10^{-4} \mathrm{~kg} \cdot \mathrm{~m}^{2} ; \ell=0.21 \mathrm{~m} ; \ell 3=0.025 \mathrm{~m} ; a=0.365 ; b=0.0022$ m (so that the position of the center of mass - excluding the final nut of length $\ell_{3}-$ is $R(x)=a x+b$ ); these are the central measured values, with the exception of $\kappa, I_{1}$ and $\ell$ which are taken one standard deviation off their central value. Also, the value $\theta_{0}=0.030 \mathrm{rad} \approx 1.7^{\circ}$ has been assumed. Even though the theoretical curve is the result of just a few trial calculations using the measured values ( $\pm$ one standard deviation) and is not a true fit, it is quite close to the measured data.

## DRAFT COPY

## Theoretical Problem 1

## Part A

A bungee jumper is attached to one end of a long elastic rope. The other end of the elastic rope is fixed to a high bridge. The jumper steps off the bridge and falls, from rest, towards the river below. He does not hit the water. The mass of the jumper is $m$, the unstretched length of the rope is $L$, the rope has a force constant (force to produce 1 m extension) of $k$ and the gravitational field strength is $g$.

You may assume that

- the jumper can be regarded as a point mass $m$ attached to the end of the rope,
- the mass of the rope is negligible compared to $m$,
- the rope obeys Hooke's law,
- air resistance can be ignored throughout the fall of the jumper.

Obtain expressions for the following and insert on the answer sheet:

- the distance $y$ dropped by the jumper before coming instantaneously to rest for the first time,
- the maximum speed $v$ attained by the jumper during this drop,
- the time $t$ taken during the drop before coming to rest for the first time.


## Part B

A heat engine operates between two identical bodies at different temperatures $T_{\mathrm{A}}$ and $T_{\mathrm{B}}\left(T_{\mathrm{A}}>T_{\mathrm{B}}\right)$, with each body having mass $m$ and constant specific heat capacity $s$. The bodies remain at constant pressure and undergo no change of phase.

1. Showing fill working, obtain an expression for the final temperature $T_{0}$ attained by the two bodies $A$ and $B$ if the heat engine extracts from the system the maximum amount of mechanical work that is theoretically possible.

Write your expression for the final temperature $T_{0}$ on the answer sheet.
2. Hence, obtain and write on the answer sheet an expression for this maximum amount of work available.

The heat engine operates between two tanks of water each of volume $2.50 \mathrm{~m}^{3}$. One tank is at 350 K and the other is at 300 K .
3. Calculate the maximum amount of mechanical energy obtainable. Insert the value on the answer sheet.

Specific heat capacity of water $=4.19 \times 10^{3} \mathrm{~J} \mathrm{~kg}^{-1} \mathrm{~K}^{-1}$
Density of water $=1.00 \times 10^{3} \mathrm{~kg} \mathrm{~m}^{-3}$

## Part C

It is assumed that when the earth was formed the isotopes ${ }^{238} \mathrm{U}$ and ${ }^{235} \mathrm{U}$ were present but not their decay products. The decays of ${ }^{238} \mathrm{U}$ and ${ }^{235} \mathrm{U}$ are used to establish the age of the earth, $T$.
a. The isotope ${ }^{238} \mathrm{U}$ decays with a half-life of $4.50 \times 10^{9}$ years. The decay products in the resulting radioactive series have half-lives short compared to this; to a first approximation their existence can be ignored. The decay series terminates in the stable lead isotope ${ }^{206} \mathrm{~Pb}$

Obtain and insert on the answer sheet an expression for the number of ${ }^{206} \mathrm{~Pb}$ atoms, denoted ${ }^{206} \mathrm{n}$, produced by radioactive decay with time t , in terms of the present number of ${ }^{238} \mathrm{U}$ atoms, denoted ${ }^{238} \mathrm{~N}$, and the half- life time of ${ }^{238} \mathrm{U}$. (You may find it helpfil to work in units of $10^{9}$ years.)
b. Similarly, ${ }^{235} \mathrm{U}$ decays with a half-ife of $0.710 \times 10^{9}$ years through a series of shorter half-ife products to give the stable isotope ${ }^{207} \mathrm{~Pb}$
... . . . . . $207 . .235_{\mathrm{KT}}$. . . . .... $235_{\mathrm{Tt}}$
c. A uranium ore, mixed with a lead ore, is analysed with a mass spectrometer. The relative concentrations of the three lead isotopes ${ }^{204} \mathrm{~Pb},{ }^{206} \mathrm{~Pb}$ and ${ }^{207} \mathrm{~Pb}$ are measured and the number of atoms are found to be in the ratios $1.00: 29.6: 22.6$ respectively. The isotope ${ }^{204} \mathrm{~Pb}$ is used for reference as it is not of radioactive origin. Analysing a pure lead ore gives ratios of $1.00: 17.9: 15.5$.

Given that the ratio ${ }^{238} \mathrm{~N}:{ }^{235} \mathrm{~N}$ is $137: 1$, derive and insert on the answer sheet an equation involving $T$.
d. Assume that $T$ is much greater than the half lives of both uranium isotopes and hence obtain an approximate value for $T$.
e. This approximate value is clearly not significantly greater than the longer half life, but can be used to obtain a much more accurate value for $T$. Hence, or otherwise, estimate a value for the age of the earth correct to within $2 \%$.

## Part D

Charge $Q$ is uniformly distributed in vacuo throughout a spherical volume of radius $R$
a. Derive expressions for the electric field strength at distance $r$ from the centre of the sphere for $r \leq R$ and $r \quad>R$.
b. Obtain an expression for the total electric energy associated with this distribution of charge.

Insert your answers to (a) and (b) on the answer sheet.

## Part E

A circular ring of thin copper wire is set rotating about a vertical diameter at a point within the Earth's magnetic field. The magnetic flux density of the Earth's magnetic field at this point is $44.5 \mu \mathrm{~T}$ directed at an angle of $64^{\circ}$ below the horizontal. Given that the density of copper is $8.90 \times 10^{3} \mathrm{~kg} \mathrm{~m}^{-3}$ and its resistivity is 1.70 $\times 10^{-8} \Omega \mathrm{~m}$, calculate how long it will take for the angular velocity of the ring to halve. Show the steps of your working and insert the value of the time on the answer sheet. This time is much longer than the time for one revolution.

You may assume that the frictional effects of the supports and air are negligible, and for the purposes of this question you should ignore self-inductance effects, although these would not be negligible.

## DRAFT COPY

## Theoretical Problem 2

a. A cathode ray tube (CRT), consisting of an electron gun and a screen, is placed within a uniform constant magnetic field of magnitude $\mathbf{B}$ such that the magnetic field is parallel to the beam axis of the gun, as shown in figure 2.1.


Figure 2.1
The electron beam emerges from the anode of the electron gun on the axis, but with a divergence of up to $5^{\circ}$ from the axis, as illustrated in figure 2.2 . In general a diffuse spot is produced on the screen, but for certain values of the magnetic field a sharply focused spot is obtained.


Figure 2.2

By considering the motion of an electron initially moving at an angle $\beta$ (where $0 \leq \beta \leq 5^{\circ}$ ) to the axis as it leaves the electron gun, and considering the components of its motion parallel and perpendicular to the axis, derive an expression for the charge to mass ratio e/m for the electron in terms of the following quantities:

- the smallest magnetic field for which a focused spot is obtained,
- the accelerating potential difference across the electron gun $V$ (note that $V<2 \mathrm{kV}$ ),
- $D$, the distance between the anode and the screen.

Write your expression in the box provided in section 2a of the answer sheet.
b. Consider another method of evaluating the charge to mass ratio of the electron. The arrangement is shown from a side view and in plan view (from above) in figure 2.3, with the direction of the magnetic field marked $\mathbf{B}$. Within this uniform magnetic field $\mathbf{B}$ are placed two brass circular plates of radius $\rho$ which are separated by a very small distance $t$. A potential difference $V$ is maintained between them. The plates are mutually parallel and co-axial, however their axis is perpendicular to the magnetic field. A photographic film, covers the inside of the curved surface of a cylinder of radius $\rho+s$, which is held co-axial with the plates. In other words, the film is at a radial distance $s$ from the edges of the plates. The entire arrangement is placed in vacuo. Note that $t$ is very much smaller than both $s$ and $\rho$.

A point source of $\beta$ particles, which emits the $\beta$ particles uniformly in all directions with a range of velocities, is placed between the centres of the plates, and the same piece of film is exposed under three different conditions:

- firstly with $B=0$, and $V=0$,
- secondly with $B=B_{0}$, and $V=V_{0}$, and
- thirdly with $B=-B_{0}$, and $V=-V{ }_{0}$;
where $V_{0}$ and $B_{0}$ are positive constants. Please note that the upper plate is positively charged when $V>0$ (negative when $V<0$ ), and that the magnetic field is in the direction defined by figure 2.5 when $B>0$ (in the opposite direction when $B<0$ ). For this part you may assume the gap is negligibly small.

Two regions of the film are labelled A and B on figure 2.3. After exposure and development, a sketch of one of these regions is given in figure 2.4.

From which region was this piece taken (on your answer sheet write A or B )? Justify your answer by showing the directions of the forces acting on the electron.
c. After exposure and development, a sketch of the film is given in figure 2.4. Measurements are made of the separation of the two outermost traces with a microscope, and this distance ( $y$ ) is also indicated for one particular angle on figure 2.4. The results are given in the table below, the angle $\phi$ being defined in figure 2.3 as the angle between the magnetic field and a line joining the centre of the plates to the point on the film.

| Angle to field /degrees | $\phi$ | 90 | 60 | 50 | 40 | 30 | 23 |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Separation $/ \mathrm{mm}$ | $y$ | 17.4 | 12.7 | 9.7 | 6.4 | 3.3 | End of trace |

Numerical values of the system parameters are given below:
$B_{0}=6.91 \mathrm{mT} V_{0}=580 \mathrm{~V} t=0.80 \mathrm{~mm} s=41.0 \mathrm{~mm}$
In addition, you may take the speed of light in vacuum to be $3.00 \times 10^{8} \mathrm{~m} \mathrm{~s}^{-1}$, and the rest mass of the electron to be $9.11 \times 10^{-31} \mathrm{~kg}$.
Determine the maximum $\beta$ particle kinetic energy observed.
Write the maximum kinetic energy as a numerical result in eV in the box on the answer sheet, section 2 c .
d. Using the information given in part (c), obtain a value for the charge to rest mass ratio of the electron. This should be done by plotting an appropriate graph on the paper provided.

Indicate algebraically the quantities being plotted on the horizontal and vertical axes both on the graph itself and on the answer sheet in the boxes provided in section 2d.

Write your value for the charge to mass ratio of the electron in the box provided on the answer sheet, section 2 d .
Please note that the answer you obtain may not agree with the accepted value because of a systematic error in the observations.

## Additional Figures

Figure 2.3

Side View:


View from above:


Figure 2.4
. . 0


## DRAFT COPY

## Theoretical Problem 3

## Part A

This part is concerned with the difficulties of detecting gravitational waves generated by astronomical events. It should be realised that the explosion of a distant supernova may produce fluctuations in the gravitational field strength at the surface of the Earth of about $10^{-19} \mathrm{~N} \mathrm{~kg}^{-1}$. A model for a gravitational wave detector (see figure 3.1) consists of two metal rods each 1 m long, held at right angles to each other. One end of each rod is polished optically flat and the other end is held rigidly. The position of one rod is adjusted so there is a minimum signal received from the photocell (see figure 3.1).


Figure 3.1

## Figure 3.1

The rods are given a short sharp longitudinal impulse by a piezoelectric device. As a result the free ends of the rods oscillate with a longitudinal displacement $\mathrm{D} x_{t}$, where

$$
\Delta x_{t}-a e^{-\mu} \cos (a t+\not p) .
$$

and $a, m, w$ and $f$ are constants.
(a) If the amplitude of the motion is reduced by $20 \%$ during a 50 s interval determine a value for $m$.
(b) Given that longitudinal wave velocity, $v=\ddot{O}(\mathrm{E} / \rho)$, determine also the lowest value for $w$, given that the rods are made of aluminium with a density $(r)$ of 2700 $\mathrm{kg}. \mathrm{~m}^{-3}$ and a Young modulus ( $E$ ) of $7.1 \times 10^{10} \mathrm{~Pa}$.
(c) It is impossible to make the rods exactly the same length so the photocell signal has a beat frequency of 0.005 Hz . What is the difference in length of the rods?
(d) For the rod of length $l$, derive an algebraic expression for the change in length, Dl , due to a change, Dg , in the gravitational field strength, $g$, in terms of $l$ and other constants of the rod material. The response of the detector to this change takes place in the direction of one of the rods.
(e) The light produced by the laser is monochromatic with a wavelength of 656 nm . If the minimum fringe shift that can be detected is $10^{-4}$ of the wavelength of the laser, what is the minimum value of $l$ necessary if such a system were to be capable of detecting variations in $g$ of $10^{-19} \mathrm{~N} \mathrm{~kg}^{-1}$ ?

## Part B

Inis part is concerned with the eftect of a gravitational field on the propagation of ught in space.
(a) A photon emitted from the surface of the Sun (mass $M$, radius $R$ ) is red-shifted. By assuming a rest-mass equivalent for the photon energy, apply Newtonian gravitational theory to show that the effective (or measured) frequency of the photon at infinity is reduced (red-shifted) by the factor ( $1-G M / R c^{2}$ ).
(b) A reduction of the photon's frequency is equivalent to an increase in its time period, or, using the photon as a standard clock, a dilation of time. In addition, it may be shown that a time dilation is always accompanied by a contraction in the unit of length by the same factor.

We will now try to study the effect that this has on the propagation of light near the Sun. Let us first define an effective refractive index $n_{r}$ at a point $r$ from the centre of the Sun. Let

$$
n_{r}=\frac{c}{c_{r}^{\prime}}
$$

where $c$ is the speed of light as measured by a co-ordinate system far away from the Sun's gravitational influence ( $r$ (B) $\boldsymbol{\sim}$ ), and $c_{r} \notin$ is the speed of light as measured by a co-ordinate system at a distance $r$ from the centre of the Sun.

Show that $n_{r}$ may be approximated to:

$$
n_{r}=1+\frac{a G M}{r c^{2}}
$$

for small $G M / r c^{2}$, where $a$ is a constant that you determine.
(c) Using this expression for $n_{r}$, calculate in radians the deflection of a light ray from its straight path as it passes the edge of the Sun.

Data:
Gravitational constant, $G=6.67 \cdot 10^{-11} \mathrm{~N} \mathrm{~m}^{2} \mathrm{~kg}^{-2}$.
Mass of Sun, $M=1.99^{\prime} 10^{30} \mathrm{~kg}$.
Radius of Sun, $R=6.95^{\prime} 10^{8} \mathrm{~m}$.
Velocity of light, $c=3.00 \cdot 10^{8} \mathrm{~m} \mathrm{~s}^{-1}$.
You may also need the following integral
$\int_{-\infty}^{\infty} \frac{d x}{\left(x^{2}+a^{2}\right)^{3 / 2}}=\frac{2}{a^{2}}$.

# 31 ${ }^{\text {st }}$ International Physics Olympiad 

## Leicester, U.K.

## Experimental Competition

Wednesday, July $12^{\text {th }}, 2000$

## Please read this first:

1. The time available is $21 / 2$ hours for each of the 2 experimental questions. Answers for your first question will be collected after $21 / 2$ hours.
2. Use only the pen issued in your back pack.
3. Use only the front side of the sheets of paper provided. Do not use the side marked with a cross.
4. Each question should be answered on separate sheets of paper.
5. For each question, in addition to the blank writing sheets where you may write, there is an answer sheet where you must summarise the results you have obtained. Numerical results should be written with as many digits as are appropriate to the given data. Do not forget to state the units
6. Write on the blank sheets of paper the results of all your measurements and whatever else you consider is required for the solution of the question and that you wish to be marked. However you should use mainly equations, numbers, symbols, graphs and diagrams. Please use as little text as possible.
7. It is absolutely essential that you enter in the boxes at the top of each sheet of paper used your Country and your student number (Student No.). In addition, on the blank sheets of paper used for each question, you should enter the number of the question (Question $N$. .), the progressive number of each sheet (Page $N$.) and the total number of blank sheets that you have used and wish to be marked for each question (Total No. of pages). It is also helpful to write the question number and the section label of the part you are answering at the beginning of each sheet of writing paper. If you use some blank sheets of paper for notes that you do not wish to be marked, put a large cross through the whole sheet and do not include it in your numbering.
8. When you have finished, arrange all sheets in proper order (for each question put answer sheets first, then used sheets in order, followed by the sheets you do not wish to be marked. Put unused sheets and the printed question at the bottom). Place the papers for each question inside the envelope labelled with the appropriate question number, and leave everything on your desk. You are not allowed to take any sheets of paper out of the room.

CDROM SPECTROMETER

## In this experiment, you are NOT expected to indicate uncertainties in your measurements.

The aim is to produce a graph showing how the conductance* of a light-dependent resistor (LDR) varies with wavelength across the visible spectrum.
$*_{\text {conductance }} G=1$ /resistance (units: siemens, $1 \mathrm{~S}=1 \Omega^{-1}$ )
There are five parts to this experiment:

- Using a concave reflection grating (made from a strip of CDROM) to produce a focused first order spectrum of the light from bulb $A$ ( 12 V 50 W tungsten filament).
- Measuring and plotting the conductance of the LDR against wavelength as it is scanned through this first order spectrum.
- Showing that the flament in bulb A behaves approximately as an ideal black body.
- Finding the temperature of the filament in bulb $A$ when it is connected to the 12 V supply.
- Correcting the graph of conductance against wavelength to take account of the energy distribution within the spectrum of light emitted by bulb $A$.


## Precautions

- Beware of hot surfaces.
- Bulb B should not be connected to any potential difference greater than 2.0 V .
- Do not use the multimeter on its resistance settings in any live circuit.


## Procedure

(a) The apparatus shown in Figure 1 has been set up so that light from bulb A falls normally on the curved grating and the LDR has been positioned in the focused first order spectrum. Move the LDR through this first-order spectrum and observe how its resistance (measured by multimeter $X$ ) changes with position.
(b) (i) Measure and record the resistance $R$ of the LDR at different positions within this first-order spectrum. Record your data in the blank table provided.
(ii) Plot a graph of the conductance $G$ of the LDR against wavelength $\lambda$ using the graph paper provided.

Note The angle $\theta$ between the direction of light of wavelength $\lambda$ in the first-order spectrum and that of the white light reflected from the grating (see Figure 1) is given by:
$\sin \theta=\lambda / d$ where $d$ is the separation of lines in the grating.
The grating has 620 lines per mm .
The graph plotted in (b)(ii) does not represent the sensitivity of the LDR to different wavelengths correctly as the emission characteristics of bulb A have not been taken into account. These characteristics are investigated in parts (c) and (d) leading to a corrected curve plotted in part (e).

- Note for part (c) that three multimeters are connected as ammeters. These should NOT be adjusted or moved. Use the fourth multimeter (aabelled X ) for all voltage measurements.
(c) If the filament of a 50 W bulb acts as a black-body radiator it can be shown that the potential difference $V$ across it should be related to the current $I$ through it by the expression:

$$
V^{3}=C I^{5} \text { where } C \text { is a constant. }
$$

Measure corresponding values of $V$ and $I$ for bulb A (in the can). The ammeter is already connected and should not be adjusted.
(i) Record your data and any calculated values in the table provided on the answer sheet.
(ii) Plot a suitable graph to show that the filament acts as a black-body radiator on the graph paper provided.
(d) To correct the graph in (b)(ii) we need to know the working temperature of the tungsten filament in bulb A . This can be found from the variation of filament resistance with temperature.

- You are provided with a graph of tungsten resistivity ( $\mu \Omega \mathrm{cm}$ ) against temperature ( K ).

If the resistance of the filament in bulb A can be found at a known temperature then its temperature when run from the 12 V supply can be found from its resistance at that operating potential difference. Unfortunately its resistance at room temperature is too small to be measured accurately with this apparatus. However, you are provided with a second smaller bulb, C , which has a larger, measurable resistance at room temperature. Bulb C can be used as an intermediary by following the procedure described below. You are also provided with a second 12 V 50 W bulb ( B ) identical to bulb A . Bulbs B and C are mounted on the board provided and connected as shown in Figure 2.
(i) Measure the resistance of bulb C when it is unlit at room temperature (use multimeter $X$, and take room temperature to be 300 K ). Record this resistance $R_{\mathrm{C} 1}$ on the answer sheet.
i. Use the circuit shown in Figure 2 to compare the filaments of bulbs B and C . Use the variable resistor to vary the current through bulb C until you can see that overlapping filaments are at the same temperature. If the small filament is cooler than the larger one it appears as a thin black loop. Measure the resistances of bulbs B and C when this condition has been reached and record their values, $R_{\mathrm{C} 2}$ and $R_{\mathrm{B}}$, on the answer sheet. Remember, the ammeters are already connected.
(iii) Use the graph of resistivity against temperature (supplied) to work out the temperature of the filaments of B and C when they are matched. Record this temperature, $T_{2} \mathrm{~V}$, on the answer sheet.
(iv) Measure the resistance of the filament in bulb A (in the can) when it is connected to the 12 V a.c. supply. Once again the ammeter is already connected and should not be adjusted. Record this value, $R_{12 \mathrm{~V}}$ on the answer sheet.
(v) Use the values for the resistance of bulb A at 2 V and 12 V and its temperature at 2 V to work out its temperature when run from the 12 V supply. Record this temperature, $T_{12 \mathrm{~V}}$ in the table on the answer sheet.

- You are provided with graphs that give the relative intensity of radiation from a black-body radiator (Planck cuves) at $2000 \mathrm{~K}, 2250 \mathrm{~K}, 2500 \mathrm{~K}$, $2750 \mathrm{~K}, 3000 \mathrm{~K}$ and 3250 K .
(e) Use these graphs and the result from (d)(v) to plot a corrected graph of LDR conductance (arbitrary units) versus wavelength using the graph paper provided. Assume that the conductance of the $\operatorname{LDR}$ at any wavelength is directly proportional to the intensity of radiation at that wavelength (This assumption is reasonable at the low intensities falling on the LDR in this experiment). Assume also that the grating diffracts light equally to all parts of the first order spectrum.

Figure 1 - Experimental arrangement for (a)


Figure 1: Detail - the grating:


Figure 1: Detail-LDR and Multimeter:


## Figure 2

a.c. power supply



Note that this diagram does not show meters

## Graph 1

Graph 1: tungsten resistivity


## Graph 2a

Graph 2(a): Planck Curves for 2000 K, $2250 \mathrm{~K}, 2500 \mathrm{~K}$



## Graph 2b

Graph 2(b): Planck Curves for 2750 K, 3000 K, 3250 K



# DRAFT COPY 

## The Magnetic Puck

July 2000

## 2.5 hours

In this experiment you $A R E$ expected to indicate uncertainties in your measurements, results and graphs

## Aim

To investigate the forces on a puck when it slides down the slope.

## Warning

Do not touch the circular flat faces of the puck or the paper surface of the slope with your hands. Use the glove provided. The faces have different coloured paper stickers for convenience but the frictional characteristics of the paper faces may be assumed to be the same.

## Timing

The sensors underneath the track trigger electronic gates in the box and the green LED will light when the puck is between the sensors. The multimeter measures the potential difference across a capacitor, which is connected to a constant-current source (whose current is proportional to the voltage of the battery) whilst the green light is on. The reading of the multimeter is therefore a measure of the time during which the puck is between the sensors. This reading can give a value for the speed of the puck in arbitrary units.

## Operating the timer

i) Press and hold down the black push button on the side of the box. This switches the electronics on.
ii) If the green light goes on slide the puck (light face up) past the lower sensor. The green light should go off
iii) The potential difference across the capacitor can be reduced to zero before the puck is released by pressing the red button for at least 10 s.
iv) The battery potential difference can be measured by connecting the multimeter across the terminals marked with the cell symbol.

## Definitions

(i) A moving body sliding down an inclined plane experiences a tangential retarding force $F$ and a normal reaction $N$. Define

$$
\xi=\frac{F}{N}
$$

(ii) When the retarding force is due to friction alone, $x$ equals $m_{S}$ and is called the dynamic coefficient of friction for the surface. It is independent of speed.
(iii) When the blue (dark) side is in contact with the plane define

$$
\xi_{d}=\frac{F_{d}}{N}
$$

where the tangential force $F_{d}$ is partly due to the surface friction and partly due to magnetic effects.
(iv) The variable $x d s$ which gives the magnetic effects only is defined by

$$
\stackrel{5}{s}-y_{s}-\mu_{s}
$$

## Important hints and advice

(i) You will find it helpful initially to investigate the behaviour of the puck qualitatively.
(ii) Think about the physics before you do a quantitative investigation. Remember to use graphical presentation where possible.
(iii) Do not attempt to take too many experimental readings unless you have plenty of time.
(iv) You are measuring the potential difference across an electrolytic capacitor. This does not behave quite like a simple air capacitor. Slow leakage of charge is normal and the potential difference will not remain completely steady.
(v) You are given one puck and one 9.0 V battery. Conserve the battery! The constant current filling the capacitor is proportional to the battery potential difference. It is therefore advisable to monitor the battery potential difference. In addition, the sensors may not be reliable if the potential difference of the battery falls below 8.4 V . If this happens, ask for another battery.
(vi) Your answer pack contains 4 sides of graph paper only. You will not be given further sheets. You may keep the puck at the end of your experiment.
(wii) Tf wnin hawe tronible nnerating the multimeters ask an invicilator

## Data

- Weight of puck $=5.84^{\prime} 10^{-2} \mathrm{~N}$
- The voltmeter reading indicates the time of travel of the puck. When the potential difference of the battery is 9.0 V then 1 V corresponds to 0.213 s
- Distance between sensors $=0.294 \mathrm{~m}$


## Experiment

Using only the apparatus provided investigate how $x d s$ depends on the speed $v_{q}$ of the puck for track inclinations $q$ to the horizontal. State on the answer sheet the algebraic equations/relations used in analysing your results and in plotting your graphs.

Suggest a quantitative model to explain your results. Use the data which you collect to justify your model.

## Question 1

## A Bungee Jumper

(a) The jumper comes to rest when
lost gravitational potential energy $=$ : stored strain energy

$$
\begin{gathered}
m g y=\frac{1}{2} k(y-L)^{2} \\
k y^{2}-2 y(k L+m g)+k L^{2}=0
\end{gathered}
$$

This is solved as a quadratic.

$$
\begin{aligned}
y= & \frac{2(k L+m g) \pm \sqrt{4(k L+m g)^{2}-4 k^{2} L^{2}}}{2 k} \\
& =\frac{k L+m g \pm \sqrt{2 m g k L+m^{2} g^{2}}}{k}
\end{aligned}
$$

Need positive root; lower position of rest (other root after initial rise).
(b) The maximum speed is attained when the acceleration is zero and forces balance; i.e. when $m g=k x$

Also kinetic energy $=$ lost potential energy - strain energy within elastic rope

$$
\begin{gathered}
\frac{1}{2} m v^{2}=m g(L+x)-\frac{1}{2} k x^{2} \\
x=\frac{m g}{k} \\
v^{2}=2 g\left(L+\frac{m g}{k}\right)-\frac{m g^{2}}{k} \\
v=\sqrt{2 g L+\frac{m g^{2}}{k}}
\end{gathered}
$$

(c) Time to come to rest $=$ time in free fall + time in SHM of rope to stop stretching

$$
\text { Length of free fall }=L=\frac{1}{2} g l_{\mathrm{f}}^{2}
$$

$$
\text { Therefore } t_{\mathrm{f}}=\sqrt{\frac{2 L}{g}}
$$

The jumper enters the SHM with free fall velocity $=g t_{\mathrm{f}}=\sqrt{2 g L}=v_{\tau}$

$$
\text { Period of SHM }=2 \pi \sqrt{\frac{m}{k}}=T
$$

We represent a full SHM cycle by
down


The jumper enters the SHM at time $\tau$ given by

$$
\tau=\frac{1}{\omega} \sin ^{-1} \frac{v_{t}}{v}=\frac{1}{\omega} \sin ^{-1} \frac{\sqrt{2 g L}}{v}
$$

Jumper comes to rest at one half cycle of the SHM at total time given by

$$
\begin{aligned}
& =t_{\mathrm{f}}+(T / 2-\tau) \\
& =\sqrt{\frac{2 I}{g}}+\pi \sqrt{\frac{m}{k}}-\frac{1}{\omega} \sin ^{-1} \frac{\sqrt{2 g L}}{v} \\
& =\sqrt{\frac{2 L}{g}}+\pi \sqrt{\frac{m}{k}} \cdot \frac{1}{\omega} \sin ^{-1} \frac{\sqrt{2 g L}}{\sqrt{2 g L+m g^{2} / k}} \\
& =\sqrt{\frac{2 L}{g}}+\sqrt{\frac{m}{k}\left\{\pi-\sin ^{-1} \frac{\sqrt{2 g L}}{\sqrt{2 g L+m g^{2} / k}}\right\}}
\end{aligned}
$$

This is the same as

$$
=\sqrt{\frac{2 L}{g}}+\sqrt{\frac{m}{k}}\left\{\frac{\pi}{2}+\cos ^{-1} \frac{\sqrt{2 g L}}{\sqrt{2 g L+m g^{2} / k}}\right\}
$$

$$
=\sqrt{\frac{2 L}{g}}+\sqrt{\frac{m}{k}} \tan ^{-1}\left\{-\sqrt{\frac{2 k L}{m g}}\right\}
$$

B Heat Engine Question


In calculating work obtainable, we assume no loss (friction etc.) in engine working.
$\Delta Q_{1}=$ energy from body A

$$
=-m s \Delta T_{1} \quad\left(\Delta T_{1}-v e\right)
$$

$\Delta Q_{2}=m s \Delta T_{2} \quad\left(\Delta T_{2}+\mathrm{ve}\right)$
(a) For maximum amount of mechanical energy assume Carnot engine

$$
\frac{\Delta Q_{1}}{T_{1}}=\frac{\Delta Q_{2}}{T_{2}} \text { throughout operation (second law) }
$$

But $\Delta Q_{1}=-m s \Delta T_{1}$ and $\Delta Q_{2}=m s \Delta T_{2}$

$$
\begin{gathered}
-m s \int_{\Gamma_{\mathrm{A}}}^{T_{0}} \frac{d T_{1}}{T_{1}}=m s \int_{T_{\mathrm{B}}}^{T_{\mathrm{B}}} \frac{d T_{2}}{T_{2}} \\
\ln \frac{T_{\mathrm{A}}}{T_{0}}=\ln \frac{T_{0}}{T_{\mathrm{B}}} \\
T_{0}^{2}=T_{\mathrm{A}} T_{\mathrm{B}} \\
T_{0}=\sqrt{T_{\mathrm{A}} T_{\mathrm{B}}}
\end{gathered}
$$

$$
\begin{aligned}
& Q_{1}=-m s \int_{T_{A}}^{T_{n}} \mathrm{~d} T_{1}=m s\left(T_{\mathrm{A}}-T_{0}\right) \\
& Q_{2}=m s \int_{T_{B}}^{r_{0}} \mathrm{~d} T_{2}=m s\left(T_{0}-T_{\mathrm{B}}\right) \\
& W=Q_{1}-Q_{2} \\
& W=m s\left(T_{\mathrm{A}}-T_{0}-T_{0}+T_{\mathrm{B}}\right)=m s\left(T_{\mathrm{A}}+T_{\mathrm{B}}-2 T_{0}\right)=m s\left(T_{\mathrm{A}}+T_{\mathrm{B}}-2 \sqrt{T_{A} T_{B}}\right) \\
& \text { or } \quad m s\left(\sqrt{T_{\mathrm{A}}}-\sqrt{T_{\mathrm{B}}}\right)^{2}
\end{aligned}
$$

(d) Numerical example:

Mass $=$ volume $\times$ density

$$
\begin{aligned}
\mathrm{W} & =2.50 \times 1.00 \times 10^{3} \times 4.19 \times 10^{3} \times(350+300-2 \sqrt{350 \times 300}) \mathrm{J} \\
& =20 \times 10^{6} \mathrm{~J} \\
& =20 \mathrm{M} .
\end{aligned}
$$

## C Radioactivity and age of the Earth

(a)

$$
\begin{aligned}
& N=N_{0} \mathrm{e}^{-\lambda} \quad N_{0}=\text { original number } \\
& n=N_{0}\left(1-\mathrm{e}^{-\lambda t}\right)
\end{aligned}
$$

Therefore $n=N e^{i t}\left(1-\mathrm{e}^{-\lambda t}\right)=N\left(\mathrm{e}^{2 \lambda}-1\right)$

So $n=N\left(2^{\prime t}-1\right)$ where $r$ is half-life
or as $\lambda=\frac{\ln 2}{T}=\frac{0.6931}{T}, \mathrm{n}=\mathrm{N}\left(e^{\frac{0.693!}{T}}-1\right)$
${ }^{206} n={ }^{228} N\left(2^{1 / 450}-1\right)$ or ${ }^{206} n={ }^{238} N\left(e^{01540 t}-1\right)$ where time $r$ is in $10^{9}$ years
(b)
${ }^{207} n={ }^{235} N\left(2^{110710}-1\right)$ or ${ }^{207} n={ }^{235} N\left(e^{09962 t}-1\right)$
(c) In mixed uranium (i.e. containing Pb of both natural and radioactive origin)
$204: 206: 207$ have proportions $\quad 1.00: 29.6: 22.6$
In pure lead (no radioactivity) $\quad 1.00: 17.9: 15.5$

Therefore for radioactively produced lead by subtraction

| $204: 206: 207$ have proportions | $1.00: 29.6: 22.6$ |
| :--- | :--- |
| In pure Icad (no radioactivity) | $1.00: 17.9: 15.5$ |

Therefore for radioactivity produced lead by subtraction
206:207
$11.7: 7.1$

Dividing equations from (a) and (b) gives

$$
\begin{aligned}
& \frac{11.7}{7.1}=137\left\{\frac{2^{T / 450}-1}{2^{T / 0.010}-1}\right\} \text { or } \frac{11.7}{7.1}=137\left\{\frac{e^{01550 T}-1}{e^{09762 T}-1}\right\} \\
& 0.0120\left\{2^{\text {rach }}-1\right\}=\left\{2^{\text {T4 40 }} \cdot 1\right\} \\
& \text { or } 0.0120\left\{e^{097027}-1\right\}=\left\{e^{0.18207}-1\right\}
\end{aligned}
$$

(d)

Assume $T \gg 4.50 \times 10^{9}$ and ignore 1 in both brackets:

$$
\begin{gathered}
0.0120\left\{2^{7 / 0.710}\right\}=\left\{2^{774.50}\right\} \text { or } 0.0120\left\{e^{0.97627}\right\}=\left\{\mathrm{e}^{0.15407}\right\} \\
0.0120=\left\{2^{7 / 450-710.710}\right\}=2^{7(0.222 \cdot 1.4884)}=2^{-1.18627} \\
T=-\frac{\log 0.0120}{\log 2 \times 1.1862}=5.38 \\
T=5.38 \times 10^{9} \text { years }
\end{gathered}
$$

$$
\text { or } 0.0120=\mathrm{e}^{-.08222 \mathrm{~T}} \quad \mathrm{~T}=\frac{\ln 0.0120}{-0.8222}=\frac{-4.4228}{-0.8222}=5.38
$$

$$
T=5.38 \times 10^{9} \text { years }
$$

(e) $T$ is not $\gg 4.50 \times 10^{9}$ years but is $>0.71 \times 10^{9}$ years

We can insert the approximate value for $T$ (call it $T^{*}==5.38 \times 10^{y}$ years) in the $2^{7 / 4.50}$ term and obtain a better value by iteration in the rapidly changing $2^{7 / 0.716}$ term). We now leave in the -1 's, although the -1 on the right-hand side has little effect and may be omitted).

Either

$$
\begin{gathered}
0.0120\left(\left(2^{T / 0.710}-1\right)=2^{T / 4.50}-1\right. \\
2^{7 / 0.710}-1=\frac{2^{1.1956}-1}{0.0120}=\frac{2.2904-1}{0.0120}=107.5 \\
T=0.710 \frac{\log 108.5}{\log 2}=4.80(0)
\end{gathered}
$$

Put T* $=4.80(0) \times 10^{9}$ years

$$
\begin{aligned}
& 2^{T .0 .710}=\frac{2^{10668}-1}{0.0120}=\frac{2.0948-1}{0.0120}=91.2 \\
& T=0.710 \frac{\log 91.2}{\log 2}=4.62(3) \\
& \text { Further iteration gives } 4.52
\end{aligned}
$$

So more accurate answer for $T$ to be in range $4.6 \times 10^{9}$ years to $4.5 \times 10^{9}$ years (either acceptable).

## D Spherical charge

(a) Charge density $=\rho=\frac{Q}{\frac{Q}{3} \pi R^{3}}$ within sphere
$x \leq R \quad$ Field at distance $x$ :

$$
E=\frac{\frac{3}{3} \pi x^{3} \rho}{4 \pi \varepsilon_{0} x^{2}}=\frac{Q x}{4 \pi \varepsilon_{0} R^{3}}
$$

$x>R \quad$ Field at distance x from the centre: $E=\frac{Q}{4 \pi \varepsilon_{0} x^{2}}$
(b) Method 1

Energy density is $\frac{1}{2} \varepsilon_{0} E^{2}$.
$x \leq R$
Energy in a thin shell of thickness $\delta x$ at radius $x$ is given by
$=\frac{1}{2} \varepsilon_{0} E^{2} 4 \pi x^{2} \delta x=\frac{1}{2} 4 \pi \varepsilon_{0} \frac{Q^{2} x^{2}}{\left(4 \pi \varepsilon_{0}\right)^{2} R^{6}} x^{2} \delta x$
Energy within the spherical volume $=\frac{1}{2} \frac{Q^{2}}{\left(4 \pi \varepsilon_{0}\right) R^{6}} \int_{x=0}^{x-R} x^{4} \mathrm{~d} x=\frac{1}{40} \frac{Q^{2}}{\pi \varepsilon_{0}} \frac{1}{R}$
$x>R$
Energy within spherical shell $=\frac{1}{2} \varepsilon_{0} E^{2} 4 \pi x^{2} \delta x=\frac{1}{2} 4 \pi \varepsilon_{0} \frac{Q^{2}}{\left(4 \pi \varepsilon_{0}\right)^{2} x^{4}} x^{2}$ is

Energy within the spherical volume for $x>R$
$=\frac{1}{2} \frac{Q^{2}}{\left(4 \pi \varepsilon_{0}\right)} \int_{i=R}^{x * \infty} \frac{1}{x^{2}} \mathrm{~d} x=\frac{1}{8} \frac{Q^{2}}{\pi \varepsilon_{0}} \frac{1}{R}$

Total energy associated with the charge distribution $=\frac{1}{40} \frac{Q^{2}}{\pi \varepsilon_{0}} \frac{1}{R}$
$+\frac{1}{8} \frac{Q^{2}}{\pi \varepsilon_{0}} \frac{1}{R}$

$$
=\frac{3}{20} \frac{Q^{2}}{\pi \varepsilon_{0}} \frac{1}{R}
$$

## Method 2

A shell with charge $4 \pi \pi^{2} \delta x \rho$ moves from $\infty$ to the surface of a sphere radius $x$ where the electric potential is

$$
\frac{\frac{4}{3} \pi x^{3} \rho}{4 \pi \varepsilon_{0} x}=\frac{x^{2} \rho}{3 \varepsilon_{0}}
$$

and will therefore gain electrical potential energy $\left(\frac{x^{2} \rho}{3 \varepsilon_{0}}\right)\left(4 \pi^{2} \rho\right) \delta x$
Total energy of complete sphere $=\int_{x=0}^{x=R} \frac{4 \pi \rho^{2} x^{4}}{3 \varepsilon_{0}} d x=\frac{4}{15} \frac{\pi \rho^{2} R^{5}}{\varepsilon_{0}}$
Putting $\mathrm{Q}=$ charge on sphere $=\frac{4}{3} \pi R^{3} \rho, \rho=-\frac{3 Q}{4 \pi R^{3}}$

So that total energy is $=\frac{4}{15} \pi\left(\frac{9 Q^{2}}{16 \pi^{2} R^{6}}\right) \frac{R^{5}}{\varepsilon}=\frac{3}{20} \frac{Q^{2}}{\pi \varepsilon_{0} R}$
(c) Binding energy $E_{\text {binding }}=E_{\text {electric }}-E_{\text {nuclear }}$

Binding energy is a negative energy
Therefore $-8.768=E_{\text {elocric }}-10.980 \mathrm{MeV}$ per nucleon
$E_{\text {electric }}=2.212 \mathrm{MeV}$ per nucleon

Radius of cobalt nucleus is given by $R=\frac{3}{20} \frac{Q^{2}}{\pi \varepsilon_{0} E_{\text {clectric }}^{\text {totat }}}$

$$
\begin{aligned}
& =\frac{3 \times 27^{2} \times\left(1.60 \times 10^{-19}\right)^{2}}{20 \times \pi \times 8.85 \times 10^{-12} \times 2.212 \times 10^{6} \times 57 \times 1.60 \times 10^{-19}} \mathrm{~m} \\
& =5.0 \times 10^{-15} \mathrm{~m}
\end{aligned}
$$

$$
\begin{aligned}
T=\frac{4 \rho d \ln 2}{B^{2}}= & \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^{3} \times 0.6931}{\left(44.5 \times 10^{-6} \times 0.4384\right)^{2}} \mathrm{~s} \\
& =1.10(2) \times 10^{6} \mathrm{~s}(=306 \mathrm{hr}=12 \text { days } 18 \mathrm{hr})
\end{aligned}
$$

## Method 1 Equating energy

Horizontal component of magnetic field $B$ inducing emf in ring:

$$
B=44.5 \times 10^{-6} \cos 64^{\circ}
$$

Magnetic flux through ring at angle $\theta=B r a^{2} \sin \theta$
where $a=$ radius of ring
Instantaneous emf $=\frac{\mathrm{d} \phi}{\mathrm{d} t}=B \pi a^{2} \frac{\mathrm{~d} \sin \omega \pi}{\mathrm{~d} t}$ where $\omega=$ angular velocity

$$
=B \pi a^{2} \omega \cos \omega r=B \pi a^{2} \omega \cos \theta
$$

R.m.s. emf over 1 revolution $=\frac{B \pi a^{2} \omega}{\sqrt{2}}$

Average resistive heating of ring $=\frac{B^{2} g^{2} a^{4} \omega^{2}}{2 R}$
Moment of inertia $=\frac{1}{2} m a^{2}$
Rotational energy $=\frac{1}{4} m a^{2} \omega^{2}$ where $m=$ mass of ring
Power producing change in $\omega=\frac{\mathrm{d}}{\mathrm{d} t}\left\{\frac{1}{4} m a^{2} \omega^{2}\right\}=$ $\frac{1}{4} m a^{2} 2 \omega \frac{\mathrm{~d} \omega}{\mathrm{~d} t}$

Equating: $\quad \frac{1}{2} m a^{2} \omega \frac{\mathrm{~d} \omega}{\mathrm{~d} t}=\frac{B^{2} \delta^{2} a^{4} \omega^{2}}{2 R}$

$$
\frac{\mathrm{d} \omega}{\omega}=\cdot \frac{B^{2} \pi^{2} a^{2}}{m R} \mathrm{~d} t
$$

If $T$ is time for angular velocity to halve,

$$
\begin{aligned}
& \int_{\omega}^{\omega / 2} \frac{\mathrm{~d} \omega}{\omega}=-\int_{0}^{\tau} \frac{B^{2} \pi^{2} a^{2}}{m R} \mathrm{~d} t \\
& \ln 2=\frac{B^{2} \mathrm{\partial}^{2} a^{2}}{m R} T
\end{aligned}
$$

But $R=\frac{20 \text { ap }}{\mathrm{A}}$ where A is cross-sectional area of copper ring

$$
m=2 \pi a d A \quad(d=\text { density })
$$

$$
\ln 2=\frac{B^{2} \partial^{2} a^{2} T}{\frac{20 a \rho}{A} 20 a d A}=\frac{B^{2} T}{4 \rho d}
$$

$$
\begin{aligned}
T=\frac{4 \rho d \ln 2}{B^{2}}= & \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^{3} \times 0.6931}{\left(44.5 \times 10^{-6} \times 0.4384\right)^{2}} \mathrm{~s} \\
& =1.10(2) \times 10^{6} \mathrm{~s}(=306 \mathrm{hr}=12 \text { days } 18 \mathrm{hr})
\end{aligned}
$$

## Method 2 Back Torque

Horizontal component of magnetic field $==B=44.5 \times 10^{-6} \cos 64^{\circ}$
Cross-section of area of ring is $A$
Radius of ring $=a$
Density of ring $=d$
Resistivity $=\rho$
$\omega=$ angular velocity ( $\omega$ positive when clockwise)
Resistance $R=\rho \frac{2 \pi a}{A}$
Mass of ring $\mathrm{m}=2 \pi a \mathrm{Ad}$
Moment of inertia $=M=\frac{1}{2} m a^{2}$
Magnetic flux through ring at angle $\theta=B \pi a^{2} \sin \theta$
Instantaneous emf $=\frac{\mathrm{d} \phi}{\mathrm{d} t}=B \pi a^{2} \frac{\mathrm{~d} \sin \omega t}{\mathrm{~d} t}=B \pi a^{2} \omega \cos \omega t=B \pi a^{2} \omega \cos \theta$
Induced current $=1=B \pi a^{2} \cos \theta t R$
Torque opposing motion $=\left(B \pi a^{2} \cos \theta\right) I=\frac{1}{R}\left(B \pi a^{2}\right)^{2} \omega \cos ^{2} \theta$
Work done in small $\delta \theta=\frac{1}{R}\left(B \pi a^{2}\right)^{2} \omega \frac{1}{2}(\cos 2 \theta+1) \delta \theta$
Average torque $=($ work done in $2 \pi$ revolution $) / 2 \pi$

$$
=\frac{1}{2 \pi R}\left(B \pi \pi^{2}\right)^{2} \omega \frac{1}{2} 2 \pi=\frac{1}{2 R}\left(B \pi \pi^{2}\right)^{2} \omega
$$

This equals $M \frac{\mathrm{~d} \omega}{\mathrm{~d} t}$ so that $M \frac{\mathrm{~d} \omega}{\mathrm{~d} t}=-\frac{B\left(\pi a^{2}\right) B\left(\pi a^{2}\right)^{1}}{(\rho / A)(2 \pi a)} \omega$

$$
\begin{gathered}
\frac{1}{2}(2 \pi a A d) a^{2} \frac{\mathrm{~d} \omega}{\mathrm{~d} t}=\frac{B^{2}\left(\pi a^{2}\right)^{2} A}{4 \rho \pi a} \omega \\
\frac{\mathrm{~d}(t)}{\mathrm{d} t}=-\frac{B^{2}}{4 \rho d} \omega \\
\int_{\omega}^{\omega / 2} \frac{\mathrm{~d} \omega}{\omega}=\int_{0}^{T} \frac{B^{2}}{4 \rho d} \mathrm{~d} t \\
\ln 2=\frac{B^{2} T}{4 \rho d} \\
T=\frac{4 \rho d \ln 2}{B^{2}}=\frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^{3} \times 0.6931}{\left(44.5 \times 10^{-6} \times 0.4384\right)^{2}} \mathrm{~s} \\
=
\end{gathered}
$$

## Question Two ~Solution

(a) Focusing occurs for one "cyclotron" orbit of the electron.

Angular velocity $\omega=\mathrm{e} B / \mathrm{m}$; so time for one orbit $T=2 \pi \mathrm{~m} / \mathrm{e} B$
Speed of electron $u=(2 \mathrm{e} \mathrm{V/m})^{1 / 2}$
Distance travelled $D=T u \cos \beta \approx T u=\left(2^{3 / 2} \pi / B\right)(V \mathrm{~m} / \mathrm{e})^{1 / 2}$
Thus charge to mass ratio $=\mathrm{e} / \mathrm{m}=8 \mathrm{~V} \times(\pi / B D)^{2}$
(b) Consider condition (ii) - Force due to electric field acts upwards

In region A force due magnetic field acts upwards as well, electron hits upper plate and does not reach the film.

In region B , force due magnetic field acts downwards, and if force is equal and opposite to the electrostatic force, there will be no unbalanced force, and electron will emerge from plates to expose film.

Piece was taken from region $B$.
(c) We require forces to balance. Electric force given by $\mathrm{e}^{Y}$ ! $t$, magnitude of magnetic force given by e $u B \sin \phi$, with $u$ the speed of the electron.

For these to balance we require $u=V / t B|\sin \phi|$

Maximum $u$ corresponds to minimum $\phi$ - at $23^{\circ}$
Therefore $u=2.687 \times 10^{8} \mathrm{~m} / \mathrm{s}=0.896 \mathrm{c}$.
Relativistic $\gamma=\left(1-v^{2} / \mathrm{c}^{2}\right)^{-1 / 2}=2.255$,
so kinetic energy of electron $=(\gamma-1) \mathrm{mc}^{2}=641 \mathrm{keV}$.
(d) After emerging from region between plates, electrons experience force due to magnetic field only. We approximate this by a vertical force, because angle of electron to horizontal remains small.

Acceleration caused by this force $a=B \mathrm{e} u \sin \phi / \gamma \mathrm{m}$
Initial horizontal speed is $u$, therefore time taken to reach the film after emerging from the region between the plates $t=s / u$.

Change in vertical displacement during this time $=y / 2=1 / 2 a(s / u)^{2}$
$y=B \mathrm{e} s^{2} \sin \phi / \gamma \mathrm{m} u$
From part (f), for electron to have emerged from plate, we also know $u=V / t B|\sin \phi|$.

Therefore we eliminate $u$ to obtain:

$$
y^{2}=(\mathrm{e} B s \sin \phi / \mathrm{m})^{2}\left\{\left(B s / \sin \phi / V^{\prime}\right)^{2}-(s / \mathrm{c})^{2}\right\}
$$

and we plot VERTICAL $\quad(y / B s \sin \phi)^{2}$
HORIZONTAL $\quad(B s t \sin \phi / \nu)^{2}$
Therefore we have a gradient $\quad(e / m)^{2}$
and a vertical-axis intercept $\quad-(\mathrm{e} s / \mathrm{mc})^{2}$
The intercept is read as $-537.7(\mathrm{C} \mathrm{s} \mathrm{/} \mathrm{~kg})^{2}$, giving $\quad \mathrm{e} / \mathrm{m}=1.70 \times 10^{11} \mathrm{C} / \mathrm{kg}$
The gradient is read as $2.826 \times 10^{22}(\mathrm{C} / \mathrm{kg})^{2}$, giving $\quad \mathrm{e} / \mathrm{m}=1.68 \times 10^{11} \mathrm{C} / \mathrm{kg}$.
a) $\quad \Delta x_{i}==a e^{-\mu t} \cos (\omega t+\phi), 0.8=e^{-50 \mu} \Rightarrow \mu=4.5 \times 10^{-3} \mathrm{~s}^{-3}$.
b) $\quad v=(E / \rho)^{1 / 2}=\left(7.1 \times 10^{10} / 2700\right)^{1 / 2}=5100 \mathrm{~m} \cdot \mathrm{~s}^{-1}$.

At fundamental $\lambda_{\text {rod }}=4 l=4 \mathrm{~m}$.
$f=5100 / 4=1.3 \times 10^{3} \mathrm{~Hz}$. $\omega=2 \pi f=8.1 \times 10^{3} \mathrm{rad} . \mathrm{s}^{-1}$.
c) $\quad v=f \lambda_{\text {rod }}, \delta \lambda_{\text {rod }} / \lambda_{\text {rod }}=(-) \delta f / f \Rightarrow \delta l / l$.
$\delta l=l .(\delta f / f)$.
[0.6]
$\delta l=1 \times\left(5.0 \times 10^{-3} / 1.3 \times 10^{3}\right)=3.8 \times 10^{-6} \mathrm{~m}$.
d) Change in gravitational force on rod at a distance $x$ from the free end $=m \Delta g$ and $m=\rho x A$, where $A$ is the cross-sectional area of the rod.
Change in stress $=m \Delta g / A=\rho x \Delta g$.
Change in strain $=\delta(d x) / d x=\rho x \Delta g / E$;
that is, $d x \rightarrow(1+\rho x \Delta g / E) d x \Rightarrow \Delta l=(\rho \Delta g / 2 E) l^{2}$.
e) At fundamental $\lambda_{\text {rod }}=4 l \Rightarrow \Delta l=\Delta \lambda_{\text {rod }} / 4$, for $\Delta \lambda_{\text {tod }}=656 \mathrm{~nm} / 10^{4} \Rightarrow \Delta l=656 \mathrm{~nm} /\left(4 \times 10^{4}\right)$.
$\Delta l=656 \mathrm{~nm} /\left(4 \times 10^{4}\right)=(\rho \Delta g 2 E) l^{2}$
[0.1]
$\Delta l=\left(2700 \times 10^{-19} / 14 \times 10^{10}\right) l^{2} \Rightarrow l=9.2 \times 10^{7} \mathrm{~m}$.
B a) $m c^{2}=h f \Rightarrow m=h f f c^{2}$,
[0.3]
$h f^{\prime \prime}=h f-G M m^{\prime} R$,
$\Rightarrow h f^{\prime}=h f^{\prime}\left(1-G M / R c^{2}\right), \therefore f^{\prime}=f\left(1-G M / R c^{2}\right)$.
b) $n_{r}=c / c\left(1-\left(i M / r c^{2}\right)^{2}\right.$,
$n_{r}=1+2\left(B M / r c^{2}\right.$, for small GM/rce ; i.e. $\alpha=2$.
c)


Diagram
By Snell's law: $n(r+\delta r) \sin \theta=n(r) \sin (\theta-\delta \xi)$,
$(n(r)+(d n / d r) \delta r) \sin \theta=n(r) \sin \theta-n(r) \cos \theta \delta \dot{\xi}$.
$(d n / d r) \delta r \sin \theta=-n(r) \cos \theta \delta \xi$.
Now $n(r)=1+2 G M / r c^{2}$, so $(d n / d r)=-2 G M / c^{2} r^{2}$,
and $\left(2 G M / c^{2} r^{2}\right) \sin \theta \delta r=n(r) \cos \theta \delta \xi$.
Hence $\delta \xi=\left(2 G M / c^{2} r^{2}\right) \tan \theta(\delta r / n) \approx\left(2 G M \tan \theta / c^{2} r^{2}\right) \delta r$.
Now $r^{2}=x^{2}+R^{2}$, so $r d r=x d x$.
$\int d \xi=\frac{2 G M}{c^{2}} \int \frac{\tan \theta d r}{r^{2}}=\frac{2 G M}{c^{2}} \int \frac{\tan \theta d d r}{r^{3}}=\frac{2 G M R}{c^{2}} \int_{-\infty}^{\infty} \frac{d x}{\left(x^{2}+R^{2}\right)^{3 / 2}}$
$\xi=\frac{4 G M}{R c^{2}}$ radians $=8.4 \times 10^{-6}$ radians.

# Theoretical Competition 

Monday, July $2^{\text {nd }}, 2001$

## Please read this first:

1. The time available is 5 hours for the theoretical competition.
2. Use only the pen provided.
3. Use only the front side of the paper.
4. Begin each part of the problem on a separate sheet.
5. For each question, in addition to the blank sheets where you may write, there is an answer form where you must summarize the results you have obtained. Numerical results should be written with as many digits as are appropriate to the given data.
6. Write on the blank sheets of paper whatever you consider is required for the solution of the question. Please use as little text as possible; express yourself primarily in equations, numbers, figures, and plots.
7. Fill in the boxes at the top of each sheet of paper used by writing your Country No and Country Code, your student number (Student No), the number of the question (Question No), the progressive number of each sheet (Page No), and the total number of blank sheets used for each question (Total No of pages). Write the question number and the section letter of the part you are answering at the top of each sheet. If you use some blank sheets of paper for notes that you do not wish to be marked, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order;

- answer form
- used sheets in order
- the sheets you do not wish to be marked
- unused sheets and the printed question

Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any sheets of paper out of the room.

## Question 1

## 1a) KLYSTRON

Klystrons are devices used for amplifying very high-frequency signals. A klystron basically consists of two identical pairs of parallel plates (cavities) separated by a distance $b$, as shown in the figure.


An electron beam with an initial speed $v_{0}$ traverses the entire system, passing through small holes in the plates. The high-frequency voltage to be amplified is applied to both pairs of plates with a certain phase difference (where period T corresponds to $2 \pi$ phase) between them, producing horizontal, alternating electric fields in the cavities. The electrons entering the input cavity when the electric field is to the right are retarded and vice versa, so that the emerging electrons form bunches at a certain distance. If the output cavity is placed at the bunching point, the electric field in this cavity will absorb power from the beam provided that its phase is appropriately chosen. Let the voltage signal be a square wave with period $T=1.0 \times 10^{-9} \mathrm{~s}$, changing between $V= \pm 0.5$ volts. The initial velocity of the electrons is $v_{0}=2.0 \times 10^{6} \mathrm{~m} / \mathrm{s}$ and the charge to mass ratio is $\mathrm{e} / \mathrm{m}=1.76 \times 10^{11}$ $\mathrm{C} / \mathrm{kg}$. The distance $\alpha$ is so small that the transit time in the cavities can be neglected. Keeping 4 significant figures, calculate;
a) the distance $b$, where the electrons bunch. Copy your result onto the answer form. [1.5 pts]
b) the necessary phase difference to be provided by the phase shifter. Copy your result onto the answer form. [1.0 pts]

## 1b) INTERMOLECULAR DISTANCE

Let $d_{L}$ and $d_{V}$ represent the average distances between molecules of water in the liquid phase and in the vapor phase, respectively. Assume that both phases are at $100^{\circ} \mathrm{C}$ and atmospheric pressure, and the vapor behaves like an ideal gas. Using the following data, calculate the ratio $d_{V} / d_{L}$ and copy your result onto the answer form. [2.5 pts]

Density of water in liquid phase: $\rho_{L}=1.0 \times 10^{3} \mathrm{~kg} / \mathrm{m}^{3}$,
Molar mass of water: $M=1.8 \times 10^{-2} \mathrm{~kg} / \mathrm{mol}$
Atmospheric pressure: $P_{a}=1.0 \times 10^{5} \mathrm{~N} / \mathrm{m}^{2}$
Gas constant: $R=8.3 \mathrm{~J} / \mathrm{mol}^{\circ} \mathrm{K}$
Avagadro's number: $N_{A}=6.0 \times 10^{23} / \mathrm{mol}$

## 1c) SIMPLE SAWTOOTH SIGNAL GENERATOR

A sawtooth voltage waveform $V_{0}$ can be obtained across the capacitor $C$ in Fig. 1. $R$ is a variable resistor, $V_{i}$ is an ideal battery, and $S G$ is a spark gap consisting of two electrodes with an adjustable distance between them. When the voltage across the electrodes exceeds the firing voltage $V_{f}$, the air between the electrodes breaks down, hence the gap becomes a short circuit and remains so until


Figure 1 the voltage across the gap becomes very small.
a) Draw the voltage waveform $V_{0}$ versus time $t$, after the switch is closed. [0.5 pts]
b) What condition must be satisfied in order to have an almost linearly varying sawtooth voltage waveform $V_{0}$ ? Copy your result onto the answer form. [0.2 pts]
c) Provided that this condition is satisfied, derive a simplified expression for the period $T$ of the waveform. Copy your result onto the answer form. [0.4 pts]
d) What should you vary ( $R$ and/or $S G$ ) to change the period only? Copy your result onto the answer form. [0.2 pts]
e) What should you vary ( $R$ and/or $S G$ ) to change the amplitude only? Copy your result onto the answer form. [0.2 pts]
f) You are given an additional, adjustable DC voltage supply. Design and draw a new circuit indicating the terminals where you would obtain the voltage waveform $V_{0}^{\prime}$ described in Fig. 2. [1.0 pts]

## 1d) ATOMIC BEAM

An atomic beam is prepared by heating a collection of atoms to a temperature $T$ and allowing them to emerge horizontally through a small hole (of atomic dimensions) of diameter $D$ in one side of the oven. Estimate the diameter of the beam after it has traveled a horizontal length $L$ along its path. The mass of an atom is $M$. Copy your result onto the answer form. [2.5 pts]


## Question 2

## BINARY STAR SYSTEM

a) It is well known that most stars form binary systems. One type of binary system consists of an ordinary star with mass $m_{0}$ and radius $R$, and a more massive, compact neutron star with mass $M$, rotating around each other. In all the following ignore the motion of the earth. Observations of such a binary system reveal the following information:

- The maximum angular displacement of the ordinary star is $\Delta \theta$, whereas that of the neutron star is $\Delta \phi$ (see Fig. 1).
- The time it takes for these maximum displacements is $\tau$.
- The radiation characteristics of the ordinary star indicate that its surface temperature is $T$ and the radiated energy incident on a unit area on earth's surface per unit time is $P$.
- The calcium line in this radiation differs from its normal wavelength $\lambda_{0}$ by an amount $\Delta \lambda$, due only to the gravitational field of the ordinary star. (For this calculation the photon can be considered to have an effective mass of $h / c \lambda$.)


Fig. 1

Find an expression for the distance $\ell$ from earth to this system, only in terms of the observed quantities and universal constants. Copy your result onto the answer form. [7 pts]
b) Assume that $M \gg m_{0}$, so that the ordinary star is basically rotating around the neutron star in a circular orbit of radius $r_{0}$. Assume that the ordinary star starts emitting gas toward the neutron star with a speed $v_{0}$, relative to the ordinary star (see Fig. 2). Assuming that the neutron star is the dominant gravitational force in this problem and neglecting the orbital changes of the ordinary star find the distance of closest approach $r_{f}$ shown in Fig. 2. Copy your result onto the answer form. [3pts]


Fig. 2

## Question 3

## MAGNETOHYDRODYNAMIC (MHD) GENERATOR

A horizontal rectangular plastic pipe of width $w$ and height $h$, which closes upon itself, is filled with mercury of resistivity $\rho$. An overpressure $P$ is produced by a turbine which drives this fluid with a constant speed $v_{0}$. The two opposite vertical walls of a section of the pipe with length $L$ are made of


The motion of a real fluid is very complex. To simplify the situation we assume the following:

- Although the fluid is viscous, its speed is uniform over the entire cross section.
- The speed of the fluid is always proportional to the net external force acting upon it.
- The fluid is incompressible.

These walls are electrically shorted externally and a uniform, magnetic field $\mathbf{B}$ is applied vertically upward only in this section. The set up is illustrated in the figure above, with the unit vectors $\hat{x}, \hat{y}$, $\hat{z}$ to be used in the solution.
a) Find the force acting on the fluid due to the magnetic field (in terms of $L, B, h, w, \rho$ and the new velocity $v$ ) [2.0 pts]
b) Derive an expression for the new speed $v$ of the fluid (in terms of $v_{0}, P, L, B$ and $\rho$ ) after the magnetic field is applied. [3.0 pts]
c) Derive an expression for the additional power that must be supplied by the turbine to increase the speed to its original value $v_{0}$. Copy your result onto the answer form. [2.0 pts]
d) Now the magnetic field is turned off and mercury is replaced by water flowing with speed $v_{0}$. An electromagnetic wave with a single frequency is sent along the section with length $L$ in the direction of the flow. The refractive index of water is $n$, and $v_{0} \ll c$. Derive an expression for the contribution of the fluid's motion to the phase difference between the waves entering and leaving section $L$. Copy your result onto the answer form. [3.0 pts]

IPhO2001 - theoretical competition

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|  |  |  |  |  |  |

## ANSWER FORM

1A
a)
$\square$
b)

Phase difference=

## 1B

$\frac{d_{V}}{d_{L}}=$

IPhO2001 - theoretical competition

| Country no | Country code | Student No. | Question No. | Page No. | Total <br> No. of pages |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |  |

1C
b)
c)
$\square$
d)
$\qquad$
e)

1D

New diameter of the beam $=$

IPhO2001 - theoretical competition

| Country no | Country code | Student No. | Question No. | Page No. | Total <br> No. of pages |
| :--- | :--- | :--- | :--- | :--- | :---: |
|  |  |  |  |  |  |

## ANSWER FORM

2a)
$\ell=$

2b)
$r_{f}=$

IPhO2001 - theoretical competition

| Country no | Country code | Student No. | Question No. | Page No. | Total <br> No. of pages |
| :--- | :--- | :--- | :--- | :--- | :---: |
|  |  |  |  |  |  |

## ANSWER FORM

3a)

3b)
$\mathrm{v}=$

3c)

Power $=$

3d)
Phase difference $=$

# Experimental Competition 

Saturday, June $30^{\text {th }}, 2001$

## Please read this first:

1. The time available is 5 hours for the experimental competition.
2. Use only the pen provided.
3. Use only the front side of the paper.
4. Begin each part of the problem on a separate sheet.
5. For each question, in addition to the blank sheets where you may write, there is an answer form where you must summarize the results you have obtained. Numerical results should be written with as many digits as are appropriate to the given data.
6. Write on the blank sheets of paper the results of all your measurements and whatever else you consider is required for the solution of the question. Please use as little text as possible; express yourself primarily in equations, numbers, figures and plots.
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## ROTATING LIQUID

This experiment consists of three basic parts:

1. investigation of the profile of the rotating liquid's surface and the determination of the acceleration due to gravity,
2. investigation of the rotating liquid as an optical system,
3. determination of the refractive index of the liquid.

When a cylindrical container filled with a liquid rotates about the vertical axis passing through its center with a uniform angular velocity $\omega$, the liquid's surface becomes parabolic (see Figure 1). At equilibrium, the tangent to the surface at the point $\mathrm{P}(x, y)$ makes an angle $\theta$ with the horizontal such that

$$
\begin{equation*}
\tan \theta=\frac{\omega^{2} x}{g} \quad \text { for }|x| \leq R \tag{1}
\end{equation*}
$$

where $R$ is the radius of the container and $g$ is the acceleration due to gravity.
It can further be shown that for $\omega<\omega_{\max }$ (where $\omega_{\max }$ is the angular speed at which the center of the rotating liquid touches the bottom of the container)

$$
\begin{equation*}
\text { at } x=x_{0}=\frac{R}{\sqrt{2}}, y\left(x_{0}\right)=h_{0} \tag{2}
\end{equation*}
$$

that is; the height of the rotating liquid is the same as if it were not rotating.
The profile of the rotating liquid's surface is a parabola defined by the equation

$$
\begin{equation*}
y=y_{0}+\frac{x^{2}}{4 C} \tag{3}
\end{equation*}
$$

where the vertex is at $\mathrm{V}\left(0, y_{0}\right)$ and the focus is at $\mathrm{F}\left(0, y_{0}+C\right)$. When optical rays parallel to the axis of symmetry (optical axis) reflect at the parabolic surface, they all focus at the point F (see Fig.1).

## Apparatus

- A cylindrical rigid plastic cup containing liquid glycerin. Millimetric scales are attached to the bottom and the sidewall of this cup.
- A turntable driven by a small dc electric motor powered by a variable voltage supply, which controls the angular velocity.
- A transparent horizontal screen on which you can put transparent or semi-transparent millimetric scales. The location of the screen can be adjusted along the vertical and horizontal directions.
- A laser pointer mounted on a stand. The position of the pointer can be adjusted. The head of the pointer can be changed.
- Additional head for the laser pointer.
- A ruler.
- A highlighter pen.
- A stopwatch. Push the left button to reset, the middle button to select the mode, and the right button to start and stop the timing.
- Transmission gratings with 500 or 1000 lines $/ \mathrm{mm}$.
- Bubble level.
- Glasses.


## IMPORTANT NOTES

- DO NOT LOOK DIRECTLY INTO THE LASER BEAM. BE AWARE THAT LASER LIGHT CAN ALSO BE DANGEROUS WHEN REFLECTED OFF A MIRROR-LIKE SURFACE. FOR YOUR OWN SAFETY USE THE GIVEN GLASSES.
- Throughout the whole experiment carefully handle the cup containing glycerin.
- The turntable has already been previously adjusted to be horizontal. Use bubble level only for horizontal alignment of the screen.
- Throughout the entire experiment you will observe several spots on the screen produced by the reflected and/or refracted beams at the various interfaces between the air, the liquid, the screen, and the cup. Be sure to make your measurements on the correct beam.
- In rotating the liquid change the speed of rotation gradually and wait for long enough times for the liquid to come into equilibrium before making any measurements.


## EXPERIMENT

## PART 1: DETERMINATION of $\boldsymbol{g}$ USING a ROTATING LIQUID [7.5 pts]

- Derive Equation 1.
- Measure the height $h_{0}$ of the liquid in the container and the inner diameter $2 R$ of the container.
- Insert the screen between the light source and the container. Measure the distance $H$ between the screen and the turntable (see Figure 2).
- Align the laser pointer such that the beam points vertically downward and hits the surface of the liquid at a distance $x_{0}=\frac{R}{\sqrt{2}}$ from the center of the container.
- Rotate the turntable slowly. Be sure that the center of the rotating liquid is not touching the bottom of the container.
- It is known that at $\mathrm{x}_{0}=\frac{R}{\sqrt{2}}$ the height of the liquid remains the same as the original height $h_{0}$, regardless of the angular speed $\omega$. Using this fact and measurements of the angle $\theta$ of the surface at $x_{0}$ for various values of $\omega$, perform an experiment to determine the gravitational acceleration $g$.
- Prepare tables of measured and calculated quantities for each $\omega$.
- Produce the necessary graph to calculate $g$.
- Calculate the value of $g$ and the experimental error in it
- Copy the values $2 R, x_{0}, h_{0}, H$ and the experimental value of $g$ and its error onto the answer form.


## PART 2: OPTICAL SYSTEM

In this part of the experiment the rotating liquid will be treated as an image forming optical system. Since the curvature of the surface varies with the angular speed of rotation, the focal distance of this optical system depends on $\omega$.

## 2a) Investigation of the focal distance [5.5 pts]

- Align the laser pointer such that the laser beam is directed vertically downward at the center of the container. Mark the point $P$ where the beam strikes the screen. Thus the line joining this point to the center of the cup is the optical axis of this system (see Figure 2).
- Since the surface of the liquid behaves like a parabolic mirror, any incident beam parallel to the optical axis will pass through the focal point $F$ on the optical axis after reflection.
- Adjust the speed of rotation to locate the focal point on the screen. Measure the angular speed of rotation $\omega$ and the distance $H$ between the screen and the turntable.
- Repeat the above steps for different $H$ values.
- Copy the measured values of $2 R$ and $h_{0}$ and the value of $\omega$ at each $H$ onto the answer form.
- With the help of an appropriate graph of your data, find the relationship between the focal length and the angular speed. Copy your result onto the answer form.


## 2b) Analysis of the "image" (what you see on the screen) [3.5 pts]

In this part of the experiment the properties of the "image" produced by this optical system will be analyzed. To do so, follow the steps given below.

- Remove the head of the laser pointer by turning it counterclockwise.
- Mount the new head (provided in an envelope) by turning it clockwise. Now your laser produces a well defined shape rather than a narrow beam.
- Adjust the position of the laser pointer so that the beam strikes at about the center of the cup almost normally.
- Put a semitransparent sheet of paper on the horizontal screen, which is placed close to the cup, such that the laser beam does not pass through the paper, but the reflected beam does.
- Observe the size and the orientation of the "image" produced by the source beam and the beam reflected from the liquid when it is not rotating.
- Start the liquid rotating, and increase the speed of rotation gradually up to the maximum attainable speed while watching the screen. As $\omega$ increases you might observe different frequency ranges over which the properties of the "image" are drastically different. To describe these observations complete the table on the answer form by adding a row to this table for each such frequency range and fill it in by using the appropriate notations explained on that page.


## PART 3: REFRACTIVE INDEX [3.5 pts]

In this part of the experiment the refractive index of the given liquid will be determined using a grating. When monochromatic light of wavelength $\lambda$ is incident normally on a diffraction grating, the maxima of the diffraction pattern are observed at angles $\alpha_{\mathrm{m}}$ given by the equation

$$
\begin{equation*}
m \lambda=d \sin \alpha_{m} \tag{4}
\end{equation*}
$$

where, $m$ is the order of diffraction and $d$ is the distance between the rulings of the grating. In this part of the experiment a diffraction grating will be used to determine the wavelength of the laser light and the refractive index of the liquid (see Figure 3).

- Use the grating to determine the wavelength of the laser pointer. Copy your result onto the answer form.
- Immerse the grating perpendicularly into the liquid at the center of the cup.
- Align the laser beam such that it enters the liquid from the sidewall of the cup and strikes the grating normally.
- Observe the diffraction pattern produced on the millimetric scale attached to the cup on the opposite side. Make any necessary distance measurements.
- Calculate the refractive index $n$ of the liquid by using your measurements. (Ignore the effect of the plastic cup on the path of the light.)
- Copy the result of your experiment onto the answer form.


Figure 1. Definitions of the bank angle $\theta$ at point $\mathrm{P}(x, y)$, the vertex V and the focus F for the parabolic surface produced by rotating the liquid, of initial height $h_{0}$ and radius $R$, at a constant angular speed $\omega$ around the $y$-axis.


Figure 2 Experimental setup for parts 1 and 2.

1. Laser pointer on a stand, 2. Transparent screen, 3. Motor, 4. Motor controller, 5. Turntable, 6. Axis of rotation, 7. Cylindrical container.


Figure 3 Top view of the grating in a liquid experiment.

1. Scaled sidewall, 2. Grating on a holder, 3. Laser pointer, 4. Cylindrical container.

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## ANSWER FORM

## 1) Determination of $\mathbf{g}$ using a rotating liquid

| $2 R$ | $x_{0}$ | $h_{0}$ | $H$ |
| :--- | :--- | :--- | :--- |
|  |  |  |  |

## Experimental value of $g$ :

2a) Investigation of the focal distance

| $2 R$ | $\mathrm{~h}_{0}$ |
| :--- | :--- |
|  |  |


| H | $\omega$ |
| :--- | :--- |
|  |  |
|  |  |
|  |  |
|  |  |

Relation between focal length and $\omega$ :

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## 2b) Analysis of the "image"

Use the appropriate notations explained below to describe what you see on the screen due to reflected beam
$\omega$ range: For the frequency ranges only approximate values are required.
Orientation (in comparison with the object beam as seen on the transparent screen):
Inverted : INV
Erect : ER

Variation of the size with increasing $\omega$ :
Increases : I
Decreases : D
No change : NC

For the frequency ranges you have found above:
Write " $\mathbf{R}$ " if the screen is above the focal point.
Write " $V$ " if the screen is below the focal point.

| $\omega$ Range | Orientation | Variation <br> of the size | "image" |
| :--- | :--- | :--- | :--- |
| $\omega=0$ |  |  |  |
|  |  |  |  |
|  |  |  |  |

## 3) Refractive index

Wavelength $=$

Experimental value for $n=$

## Solution

## Part 1a

$v_{\text {ret }}=\sqrt{v_{0}^{2}-2(e / \mathrm{m}) V}=1.956 \times 10^{6} \mathrm{~m} / \mathrm{s}$
$v_{a c c}=\sqrt{v_{0}^{2}+2(e / \mathrm{m}) V}=2.044 \times 10^{6} \mathrm{~m} / \mathrm{s}$
$x_{\text {ret }}=v_{\text {ret }}, \quad \mathrm{x}_{\text {acc }}=v_{\text {acc }}(t-T / 2)$
$x_{\text {ret }}=x_{\text {acc }} \rightarrow \mathrm{t}_{\text {bunch }}=\frac{v_{\text {acc }} T}{2\left(v_{a c c}-v_{\text {ret }}\right)}=11.61 T$
$b=v_{\text {ret }} t_{\text {bunch }}=2.272 \times 10^{-2} \mathrm{~m}$.
b. The phase difference:
$\Delta \varphi= \pm\left(\frac{t_{\text {bunch }}}{T}-n\right) 2 \pi= \pm 0.61 \times 2 \pi= \pm 220^{\circ}$.
(1.0 pts)

OR
$\Delta \varphi= \pm 140^{\circ}$

## Part 1b

$$
\begin{equation*}
\rho_{L}=n_{L} \frac{M}{N_{A}} \tag{0.3pts}
\end{equation*}
$$

where $n_{L}$ is the number of molecules per cubic meter in the liquid phase Average distance between the molecules of water in the liquid phase:

$$
\begin{equation*}
d_{L}=\left(n_{L}\right)^{-1 / 3}=\left(\frac{M}{\rho_{L} N_{A}}\right)^{\frac{1}{3}} \tag{0.2pts}
\end{equation*}
$$

$\mathrm{P}_{\mathrm{a}} \mathrm{V}=\mathrm{nRT}$,
where n is the number of moles

$$
P_{a}=\frac{n M}{V} \frac{R T}{M}=\rho_{V} \frac{R T}{M}=\frac{n_{V} M}{N_{A}} \frac{R T}{M}
$$

where $\mathrm{n}_{\mathrm{v}}$ is the number of molecules per cubic meter in the vapor phase.

$$
\begin{align*}
& d_{V}=\left(n_{V}\right)^{-1 / 3}=\left(\frac{R T}{P_{a} N_{A}}\right)^{\frac{1}{3}}  \tag{0.2pts}\\
& \frac{d_{V}}{d_{L}}=\left(\frac{R T \rho_{L}}{P_{a} M}\right)^{\frac{1}{3}}=12 \tag{0.3pts}
\end{align*}
$$

## Part 1c

a.

b. $\quad V_{i} \gg V_{f}$ ( 0.2 pts )
c. $\underset{\text { If }}{\left.\mathrm{V}_{\mathrm{f}}=\mathrm{V}_{\mathrm{i}}\left(1-\mathrm{e}^{-\mathrm{T} / \mathrm{RC}}\right), ~()^{2}\right)}$
(0.2 pts)
$\mathrm{V}_{\mathrm{i}} \gg \mathrm{V}_{\mathrm{f}}, \quad \quad \mathrm{T} / \mathrm{RC} \ll 1, \quad \quad e^{-T / R C} \approx 1-(T / R C)$
then
$T=\left(V_{f} / V_{i}\right) R C$
(0.2 pts)
d. $R$
e. $S G$ and $R$
f. Correct circuit
(0.2 pts)
(0.2 pts) (0.4 pts)

$V_{0}^{\prime}$
(0.3 pts)
$V_{i}^{\prime}-V_{f}$ with the correct polarity
(0.3 pts)

Total
(1.0 pts)

## Part 1d

As the beam passes through a hole of diameter $D$ the resulting uncertainty in the $y$ component of the momentum;

$$
\begin{equation*}
\Delta p_{y} \approx \frac{\hbar}{D} \tag{0.6pts}
\end{equation*}
$$

and the corresponding velocity component;

$$
\begin{equation*}
\Delta v_{y} \approx \frac{\hbar}{M D} \tag{0.4pts}
\end{equation*}
$$

Diameter of the beam grows larger than the diameter of the hole by an amount $\Delta D=\Delta v_{y} . t$,
where $t$ is the time of travel.
If the oven temperature is T, a typical atom leaves the hole with kinetic energy

$$
\begin{align*}
& K E=\frac{1}{2} M v^{2}=\frac{3}{2} k T  \tag{0.4pts}\\
& v=\sqrt{\frac{3 k T}{M}} \tag{0.2pts}
\end{align*}
$$

Beam travels the horizontal distance $L$ at speed $v$ in time

$$
\begin{align*}
& t=\frac{L}{v}, \text { so }  \tag{0.2pts}\\
& \Delta D=t \Delta v_{y} \approx \frac{L}{v} \frac{\hbar}{M D}=\frac{L \hbar}{M D \sqrt{\frac{3 k T}{M}}}=\frac{L \hbar}{D \sqrt{3 M k T}} \tag{0.4pts}
\end{align*}
$$

Hence the new diameter after a distance $L$ will be;

$$
\begin{equation*}
\mathrm{D}_{\mathrm{new}}=\mathrm{D}+\frac{L \hbar}{D \sqrt{3 M k T}} \tag{0.1pts}
\end{equation*}
$$

## Part 2a

The total energy radiated per second $=4 \pi \mathrm{R}^{2} \sigma T^{4}$, where $\sigma$ is the Stephan-Boltzmann constant. The energy incident on a unit area on earth per second is;

$$
\begin{equation*}
P=\frac{4 \pi R^{2} \sigma T^{4}}{4 \pi \ell^{2}} \text { yielding, } R=\left(P / \sigma T^{4}\right)^{1 / 2} \ell(1) \tag{0.8pts}
\end{equation*}
$$

The energy of a photon is $\mathrm{hf}=\mathrm{hc} / \lambda$. The equivalent mass of a photon is $\mathrm{h} / \mathrm{c} \lambda$. Conservation of photon energy:

$$
\begin{equation*}
\frac{h c}{\lambda_{0}}-\frac{G m_{0}}{R} \cdot \frac{h}{c \lambda_{0}}=\frac{h c}{\lambda} \tag{0.8pts}
\end{equation*}
$$

yielding

$$
R=\frac{G m_{0}\left(\lambda_{0}+\Delta \lambda\right)}{c^{2} \Delta \lambda}
$$

and (2) yields,

$$
\begin{equation*}
m_{0}=\frac{c^{2} \Delta \lambda\left(P / \sigma T^{4}\right)^{1 / 2}}{G\left(\lambda_{0}+\Delta \lambda\right)} \ell \tag{0.2pts}
\end{equation*}
$$

The stars are rotating around the center of mass with equal angular speeds:

$$
\begin{equation*}
\omega=(2 \pi / 2 \tau)=\pi / \tau(4) \tag{0.2pts}
\end{equation*}
$$

The equilibrium conditions for the stars are;

$$
\begin{equation*}
\frac{G M m_{0}}{\left(r_{1}+r_{2}\right)^{2}}=m_{0} r_{1} \omega^{2}=M r_{2} \omega^{2} \tag{0.8pts}
\end{equation*}
$$

with

$$
\begin{equation*}
r_{1}=\ell \frac{\Delta \theta}{2}, r_{2}=\ell \frac{\Delta \phi}{2} \text { (6) } \tag{0.4pts}
\end{equation*}
$$

Substituting (3), (4) and (6) into (5) yields

$$
\begin{equation*}
\ell=\left(\frac{8 c^{2} \Delta \lambda\left(P / \sigma T^{4}\right)^{1 / 2}}{\Delta \phi(\pi / \tau)^{2}\left(\lambda_{0}+\Delta \lambda\right)(\Delta \theta+\Delta \phi)^{2}}\right)^{1 / 2} \tag{0.8pts}
\end{equation*}
$$

## Part 2b

Conservation of angular momentum for the ordinary star;

$$
\begin{equation*}
m r^{2} \omega=m_{0} r_{0}^{2} \omega_{0} \tag{0.6pts.}
\end{equation*}
$$

Conservation of angular momentum for dm :

$$
\begin{equation*}
r^{2} \omega d m=r_{f}^{2} \omega_{f} d m \tag{0.6pts}
\end{equation*}
$$

where $\omega_{\mathrm{f}}$ is the angular velocity of the ring. Equilibrium in the original state yields,

$$
\begin{equation*}
\omega_{0}=\left(\frac{G M}{r_{0}^{3}}\right)^{1 / 2} \tag{0.8pts}
\end{equation*}
$$

and (7), (8) and (9) give,

$$
\begin{equation*}
\omega=\frac{m_{0} r_{0}}{m r^{2}}\left(\frac{G M}{r_{0}}\right)^{1 / 2}, \omega_{f}=\frac{m_{0} r_{0}}{m r_{f}^{2}}\left(\frac{G M}{r_{0}}\right)^{1 / 2} \tag{0.4pts}
\end{equation*}
$$

Conservation of energy for dm;

$$
\begin{equation*}
\frac{1}{2} d m\left(v_{0}^{2}+r^{2} \omega^{2}\right)-\frac{G M \mathrm{dm}}{\mathrm{r}}=\frac{1}{2} d m r_{f}^{2} \omega_{f}^{2}-\frac{G M \mathrm{dm}}{\mathrm{r}_{\mathrm{f}}}(11) \tag{1.2pts}
\end{equation*}
$$

Substituting (10);

$$
v_{0}^{2}+\frac{m_{0}^{2} r_{0} G M}{m^{2}}\left(\frac{1}{r^{2}}-\frac{1}{r_{f}^{2}}\right)-2 G M\left(\frac{1}{r}-\frac{1}{r_{f}}\right)=0
$$

Since $r_{0} \gg r_{f}$, if $r>r_{0}, r^{-1}$ and $r^{-2}$ terms can be neglected. Hence,

$$
\begin{equation*}
r_{f}=\frac{G M}{v_{0}^{2}}\left(\left(1+\frac{m_{0}^{2} r_{0} v_{0}^{2}}{G M m^{2}}\right)^{1 / 2}-1\right) \tag{0.8pts}
\end{equation*}
$$

To show that $r>r_{0}$ change in the linear momentum of the ordinary star in its reference frame:

$$
\begin{equation*}
-\frac{G M m}{r^{2}}+m r \omega^{2}-m \frac{d v_{r}}{d t}=-v_{0} \frac{d m_{g a s}}{d t} \tag{0.8pts}
\end{equation*}
$$

and (13) implies the existence of an outward force initially and hence $r$ starts growing. Using (7) one can write

$$
\begin{equation*}
m r \omega^{2}=\frac{m_{0}^{2} r_{0}^{4} \omega_{0}^{2}}{m r^{3}} \tag{0.4pts}
\end{equation*}
$$

Hence, $\frac{\text { Gravitational force }}{\text { Centrifugal force }} \alpha \mathrm{m}^{2} r$.
where $m$ is definitely decreasing. If $r$ starts decreasing at some time also, this ratio starts decreasing, which is a contradiction.

So $r>r_{0}$.

## Part 3a



The net force on a charged particle must be zero in the steady state

$$
\begin{aligned}
& \vec{F}=0=q \vec{E}+q \vec{v} x \vec{B} \\
& \vec{E}=-\vec{v} x \vec{B}=v B \hat{y} \\
& V_{H}=v B w \\
& I=\frac{V_{H}}{R}=\frac{V_{H}}{\frac{\rho w}{L h}}=\frac{v B w L h}{\rho w}=\frac{v B L h}{\rho}, \text { direction: }-\hat{y} \\
& \vec{F}=I \vec{\ell} x \vec{B}=\frac{v B^{2} L h w}{\rho}, \text { direction: }(-\hat{y} \times \hat{z}=-\hat{x})
\end{aligned}
$$

Force is in the -x direction
This creates a back pressure $\mathrm{P}_{\mathrm{b}}$

$$
\begin{align*}
& P_{b}=\frac{v B^{2} L h w}{\rho h w}=\frac{v B^{2} L}{\rho}  \tag{0.6pts}\\
& \mathrm{~F}_{\text {net }}=\left(\mathrm{P}-\mathrm{P}_{\mathrm{b}}\right) \mathrm{hw}, \\
& \mathrm{v}=\alpha \mathrm{F}_{\text {net }} \quad(0.4 \mathrm{pts}) \\
& \mathrm{v}=\alpha\left(\mathrm{P}-\mathrm{P}_{\mathrm{b}}\right) \mathrm{hw}=\alpha\left(P-\frac{v B^{2} L}{\rho}\right) \frac{v_{0}}{\alpha P}=v_{0}-\frac{v v_{0} B^{2} L}{P \rho} \\
& v\left(1+\frac{v_{0} B^{2} L}{P \rho}\right)=v_{0} \\
& v=v_{0}\left(1+\frac{v_{0} B^{2} L}{P \rho}\right)^{-1} \\
& v=v_{0} \frac{P \mathrm{pts})}{P \rho+v_{0} B^{2} L}
\end{align*}
$$

## Part 3b

From conservation of energy:
$\Delta$ Power $=V_{H} I=\frac{v_{0}^{2} B^{2} w h L}{\rho}$
or,
to recover $\mathrm{v}_{0}$ the pump must supply an additional pressure $\Delta \mathrm{P}=\mathrm{P}_{\mathrm{b}}$
$\Delta$ Power $=\Delta P h w v_{0}=P_{b} h w v_{0}=\frac{v_{0}^{2} B^{2} w h L}{\rho}$

## Part 3c

1. $u=\frac{c}{n} \quad u^{\prime}=\frac{\frac{c}{n}+v}{1+\frac{c}{n} \frac{v}{c^{2}}}=\frac{\frac{c}{n}+v}{1+\frac{v}{c n}}$

For small $\mathrm{v}(\mathrm{v} \ll \mathrm{c})$;
neglect the terms containing $\frac{v^{2}}{c^{2}}$ in the expansion of $\left(1+\frac{v}{c n}\right)^{-1}$
$u^{\prime}=\left(\frac{c}{n}+v\right) \frac{1}{1+\frac{v}{c n}} \approx\left(\frac{c}{n}+v\right)\left(1-\frac{v}{c n}\right) \approx \frac{c}{n}+v\left(1-\frac{1}{n^{2}}\right)$
$\Delta u=u^{\prime}-u \approx v\left(1-\frac{1}{n^{2}}\right)$
$\Delta \phi=2 \pi f \Delta T, T=\frac{L}{u}, \Delta \mathrm{~T}=\frac{\Delta \mathrm{u}}{\mathrm{u}^{2}} L \approx \frac{L v}{c^{2}}\left(n^{2}-1\right)$
$v=v_{0}$ so that, $\Delta \phi=2 \pi f \frac{L}{c^{2}}\left(n^{2}-1\right) v_{0}$
2. $\Delta \phi=2 \pi f \frac{L}{c^{2}}\left(n^{2}-1\right) v_{0}$
a phase of $\pi / 36$ results in
(0.4 pts)
$v_{0}=\frac{c^{2}}{72 L\left(n^{2}-1\right) f}$
$v_{0}=\frac{9 \times 10^{16}}{72 \times 10^{-1} \times(2.56-1) \times 25}=3.2 \times 10^{14} \mathrm{~m} / \mathrm{s}$ which is not physical.
(0.4 pts)
3. For $v=20 \mathrm{~m} / \mathrm{s}, \mathrm{f} \approx 4 \times 10^{14} \mathrm{~Hz}$. But for this value of f , skin depth is about 25 nm . This means that amplitude of the signal reaching the end of the tube is practically zero. Therefore mercury should be replaced with water.

On the other hand if water is used instead of mercury, at $25 \mathrm{~Hz} \delta \approx 3 \times 10^{5}$ m . Signal reaches to the end but $\mathrm{v} \approx 6 \times 10^{14} \mathrm{~m} / \mathrm{s}$, is still nonphysical. Therefore frequency should be readjusted.
(0.6 pts)

For $\mathrm{v}=20 \mathrm{~m} / \mathrm{s}$ electromagnetic wave of $\mathrm{f} \approx 8 \times 10^{14} \mathrm{~Hz}$ has a skin depth of about $\delta \approx 5.6 \mathrm{~cm}$ in water and the emerging wave is out of phase by $\pi / 36$ with respect to the incident wave. (The amplitude of the wave reaching to the end of the section is about $17 \%$ of the incident amplitude). ( 0.6 pts)

Therefore mercury should be replaced with water and frequency should be adjusted to $\mathrm{f} \approx 8 \times 10^{14} \mathrm{~Hz}$. The correct choice is (iii)
(0.2 pts)

## Solution

## Part 1

## Theory:

Consider a small mass $m$ of the liquid at the surface (Figure 4). At dynamic equilibrium

$$
\mathrm{N} \cos \theta=\mathrm{mg}
$$

and

$$
N \sin \theta=m w^{2} x
$$

Therefore:

$$
\tan \theta=\frac{w^{2} x}{g} .
$$



The profile of the liquid surface can be found as follows:

$$
\tan \theta=\frac{d y}{d x}, \quad \frac{d y}{d x}=\frac{w^{2} x}{g}
$$

so that

$$
y=\frac{w^{2} x^{2}}{2 g}+y_{0}
$$

where $y_{0}$ is the height at $x=0$.
At a certain point $x=x_{0}$, height of the liquid $h_{0}$ would be the same as if it not rotating. In this case,

$$
\begin{equation*}
h_{0}=y_{0}+\frac{w^{2} x_{0}^{2}}{2 g} \tag{1}
\end{equation*}
$$

and,

$$
x_{0}^{2}=\frac{2 g\left(h_{0}-y_{0}\right)}{w^{2}} .
$$

Since the volume of the liquid is constant,

$$
\begin{align*}
& \pi R^{2} h_{0}=\int y(2 \pi x d x)=2 \pi \int\left(y_{0}+\frac{w^{2} x^{2}}{2 g}\right) x d x \\
& y_{0}=h_{0}-\frac{w^{2} R^{2}}{4 g} \tag{2}
\end{align*}
$$

From Eq. 1 and Eq. 2 one obtains

$$
x_{0}=\frac{R}{\sqrt{2}} .
$$

## Experiment:

| $2 R(\mathrm{~mm})$ | $x_{0}(\mathrm{~mm})$ | $h_{0}(\mathrm{~mm})$ | $H(\mathrm{~mm})$ |
| :---: | :---: | :---: | :---: |
| 145 | 51 | 30 | 160 |

$\mathrm{H}-\mathrm{h}_{0}=130 \mathrm{~mm}$
Measure 10T at small speeds and measure 15T-20T at high speeds.
Use $\quad \tan (2 \theta)=\frac{x}{H-h_{0}}$
and $\quad w=\frac{2 \pi}{T}$.


| $\mathrm{h}_{0}(\mathrm{~mm})$ | $\mathrm{H}(\mathrm{mm})$ | $\mathrm{H}-\mathrm{h}_{0}(\mathrm{~mm})$ |
| :---: | :---: | :---: |
| 30 | 160 | 130 |


| $\mathrm{x}(\mathrm{mm})$ | 10T(s) | w(rad/s) | $\tan (2 \theta)$ | $\theta(\mathrm{rad})$ | $\theta$ (deg) | $\tan (\theta)$ | $\mathrm{w}^{2}(\mathrm{rad} / \mathrm{s})^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 11 | 21.34 | 2.94 | 0.08 | 0.04 | 2.4 | 0.04 | 8.67 |
| 20 | 15.80 | 3.98 | 0.15 | 0.08 | 4.4 | 0.08 | 15.81 |
| 26 | 14.22 | 4.42 | 0.20 | 0.10 | 5.7 | 0.10 | 19.52 |
| 30 | 12.99 | 4.84 | 0.23 | 0.11 | 6.5 | 0.11 | 23.40 |
| 40 | 11.74 | 5.35 | 0.31 | 0.15 | 8.6 | 0.15 | 28.64 |
| 51 | 10.45 | 6.01 | 0.39 | 0.19 | 10.7 | 0.19 | 36.15 |
| 56 | 9.90 | 6.35 | 0.43 | 0.20 | 11.7 | 0.21 | 40.28 |
| 65 | 9.40 | 6.68 | 0.50 | 0.23 | 13.3 | 0.24 | 44.68 |
| 70 | 9.08 | 6.92 | 0.54 | 0.25 | 14.2 | 0.25 | 47.88 |
| 85 | 8.39 | 7.49 | 0.65 | 0.29 | 16.6 | 0.30 | 56.08 |
| 100 | 7.71 | 8.15 | 0.77 | 0.33 | 18.8 | 0.34 | 66.41 |
| 112 | 7.43 | 8.46 | 0.86 | 0.36 | 20.4 | 0.37 | 71.51 |
| 132 | 7.00 | 8.98 | 1.02 | 0.40 | 22.7 | 0.42 | 80.57 |
| 61.4 | 11.19 | 6.20 | 0.47 | 0.21 | 11.98 | 0.21 | 41.51 |

The last line is for error calculation only.
The slope of the Figure 5 is $0.0052(\mathrm{~s} / \mathrm{rad})^{2}$ which gives

$$
g=\frac{x_{0}}{\text { slope }}=\frac{5.1}{0.0052}=980 \mathrm{~cm} / \mathrm{s}^{2} .
$$



## Error Calculation (possible methods):

$$
\begin{aligned}
& g=\frac{w^{2} x_{0}}{\tan \theta} \\
& \frac{\Delta g}{g}=\sqrt{4\left(\frac{\Delta w}{w}\right)^{2}+\left(\frac{\Delta x_{0}}{x_{0}}\right)^{2}+\left(\frac{\Delta(\tan \theta)}{\tan \theta}\right)^{2}} \frac{\Delta w}{w}=\frac{\Delta T}{T} \\
& \frac{\Delta(\tan \theta)}{\tan \theta} \approx \frac{\Delta \theta}{\theta}
\end{aligned}
$$

(since from the table $\tan \theta \cong \theta$ )

$$
\begin{aligned}
& \theta \approx \frac{x}{H-h_{0}}, \frac{\Delta \theta}{\theta}=\sqrt{\left(\frac{\Delta x}{x}\right)^{2}+\left(\frac{\Delta H+\Delta h_{0}}{H-h_{0}}\right)^{2}} \\
& \frac{\Delta g}{g}=\sqrt{4\left(\frac{\Delta T}{T}\right)^{2}+\left(\frac{\Delta x_{0}}{x_{0}}\right)^{2}+\left(\frac{\Delta x}{x}\right)^{2}+\left(\frac{\Delta H+\Delta h_{0}}{H-h_{0}}\right)^{2}}
\end{aligned}
$$

Using the values $\mathrm{H}=160 \mathrm{~mm}, \Delta \mathrm{H}=1 \mathrm{~mm}, \mathrm{~h}_{0}=30 \mathrm{~mm}, \Delta \mathrm{~h}_{0}=1 \mathrm{~mm}, \mathrm{x}_{\mathrm{av}}=61.4 \mathrm{~mm}, \Delta \mathrm{x}_{\mathrm{av}}=1 \mathrm{~mm}$, $\mathrm{T}_{\mathrm{av}}=1.1 \mathrm{~s}, \Delta \mathrm{~T}=0.01 \mathrm{~s}, \mathrm{x}_{0}=51 \mathrm{~mm}, \Delta \mathrm{x}_{0}=1 \mathrm{~mm}$ one obtains
$g=980 \pm 34 \mathrm{~cm} / \mathrm{s}^{2}$

- Note that from the method of least squares one obtains the following results:
$\mathrm{g}=982 \mathrm{~cm} / \mathrm{s}^{2}$ with a standard deviation of $\sigma=33 \mathrm{~cm} / \mathrm{s}^{2}$
- From the linear regression of the data slope $\tan \theta \mathrm{vs} \mathrm{w}^{2}$ is found to be 0.052 with a standard error of $5.14 \times 10^{-5}$, therefore:

$$
\frac{\Delta g}{g}=\sqrt{\left(\frac{\Delta(\text { slope })}{\text { slope }}\right)^{2}+\left(\frac{\Delta x_{0}}{x_{0}}\right)^{2}}=0.02
$$

$g=980 \pm 20 \mathrm{~cm} / \mathrm{s}^{2}$

## Part 2a

| $H(\mathrm{~mm})$ | $10 \mathrm{~T}(\mathrm{~s})$ | $\mathrm{w}(\mathrm{rad} / \mathrm{s})$ | $\ln w$ | $\mathrm{H}-\mathrm{h}_{0}(\mathrm{~mm})$ | $\ln \left(\mathrm{H}-\mathrm{h}_{0}\right)$ |
| ---: | ---: | ---: | :---: | ---: | ---: |
| 158 | 10.31 | 6.09 | 0.784921 | 128 | 2.107 |
| 209 | 13.19 | 4.76 | 0.677935 | 179 | 2.253 |
| 190 | 11.70 | 5.37 | 0.729994 | 160 | 2.204 |
| 150 | 9.80 | 6.41 | 0.806954 | 120 | 2.079 |
| 129 | 9.21 | 6.82 | 0.83392 | 99 | 1.996 |
| 119 | 8.75 | 7.18 | 0.856172 | 89 | 1.949 |
| 110 | 8.10 | 7.76 | 0.889695 | 80 | 1.903 |



Figure 3.

Thus the focal length depends on was

$$
f=A w^{n},
$$

and
n ~ -1.7.

The plot of $\mathrm{H}-h_{0}$ vs. $1 / \mathrm{w}^{2}$ is also acceptable as a correct plot.

## Part 2b

| $\omega$ Range(rad/s) | Orientation | Variation of the size | Image |
| :---: | :---: | :---: | :---: |
| $\omega=0$ | ER |  | V |
| $\begin{aligned} & 0<\omega<8.2^{*} \\ & 0<\omega<6.3^{* *} \end{aligned}$ | ER | D | V |
| $\begin{aligned} & 8.2<\omega<14.6^{*} \\ & 6.3<\omega<14.0^{* *} \end{aligned}$ | INV | 1 | R |
| $\begin{aligned} & 14.6<\omega<\omega_{\max }{ }^{*} \\ & 14.0<\omega<\omega_{\max }{ }^{* *} \end{aligned}$ | ER | NC | V |

* for $\mathrm{H}=110 \mathrm{~mm}$
** for $\mathrm{H}=240 \mathrm{~mm}$
$\omega$ values depend on the initial values of $\mathrm{H}, \mathrm{h}_{0}$, etc.
Note that measurements only at one H value are required from the students.


## Part 3

## Measurement of wavelength

Both the grating and the screen are in air. Normal incidence.
Screen to grating distance : L
Distance between the diffraction spots seen on the screen : x
Order of diffraction
: m

- $L=225 \mathrm{~mm}, \quad \mathrm{x}_{\mathrm{av}}=77 \mathrm{~mm} \quad$ for $\mathrm{m}= \pm 1 \quad \mathrm{~d}=1 / 500 \mathrm{~mm}$
$\tan \alpha=\frac{x_{a v}}{L}=\frac{77}{225}$
$\lambda=\frac{1}{500} \sin \alpha=647 \mathrm{~nm}$
- $L=128 \mathrm{~mm}, \quad \mathrm{X}_{\mathrm{av}}=44 \mathrm{~mm} \quad$ for $\mathrm{m}= \pm 1, \quad \mathrm{~d}=1 / 500 \mathrm{~mm}$ $\tan \alpha=\frac{44}{128}$
$\lambda=\frac{1}{500} \sin \alpha=650 \mathrm{~nm}$
- $L=128 \mathrm{~mm}, \quad x_{a v}=111 \mathrm{~mm} \quad$ for $m= \pm 2, \quad d=1 / 500 \mathrm{~mm}$ $\tan \alpha=\frac{111}{128}$

$$
\lambda=\frac{1}{2 x 500} \sin \alpha=655 \mathrm{~nm}
$$

The average value of $\lambda$ is $\lambda_{\mathrm{av}}=651 \mathrm{~nm}$.

## Measurement of refractive index

$2 R=145 \mathrm{~mm}$
Distance between the spots measured on the curved screen $=\mathrm{R} \alpha$
$\begin{array}{llll} & \mathrm{R} \alpha_{\mathrm{av}}=17 \mathrm{~mm} & \text { for } \mathrm{m}= \pm 1 & \alpha_{\mathrm{av}}=0.234 \mathrm{rad} \\ \text { using } & n=\frac{m \lambda}{d \sin (\alpha)}, & \text { one obtains } & \mathrm{n}=1.40\end{array}$
If the curvature of the screen is neglected:

$$
\begin{aligned}
& \tan \alpha=\frac{17}{72.5} \\
& \alpha=13.20^{\circ} \\
& n=\frac{\lambda}{d \sin (\alpha)}=\frac{651(\mathrm{~nm})}{\frac{1}{500}(\mathrm{~mm}) \times 10^{6} \sin (\alpha)}=1.43
\end{aligned}
$$

## Grading Scheme for Experimental Competition

## Part 1

7.5 pts

- Derivation of Equation 1
1.0 pts
- Calculation of $\omega$ using period measurements
1.0 pts

At low speeds 10T is OK
At high speeds 20T is expected -0.2 pts
Missing units -0.2 pts

- Calculation of $\tan 2 \theta, \tan \theta$ at each $\omega \quad 1.0$ pts

Calculation of $\tan 2 \theta \quad \overline{0.5 \mathrm{pts}}$
Calculation of $\tan \theta \quad 0.5 \mathrm{pts}$

- Plot of $\tan \theta$ vs $\omega^{2} \quad 1.5$ pts

Axes with labels and units $\quad 0.4 \mathrm{pts}$
Drawing best fit line 0.5 pts
At least 6 different data in a wide range of $\omega \quad 0.6$ pts
No. of measurements 5: -0.2 pts
No of measurements 4: -0.4 pts
No of measurements 3 or less: -0.6 pts

- Calculations $\quad 2.0$ pts
calculation of slope with unit $\quad \frac{2.0 \mathrm{pts}}{1.0 \mathrm{pts}}$
calculation of $g \quad 1.0$ pts
FULL credit for
$9.3<g<10.3 \mathrm{~m} / \mathrm{s}^{2}$ ( $\pm 5 \%$ error)
For $g$ values credits to be subtracted from the total credit of 7.5:
$10.3<g<10.5 \mathrm{~m} / \mathrm{s}^{2}, \quad 9.1<\mathrm{g}<9.3 \mathrm{~m} / \mathrm{s}^{2}$
-0.5 pts
$8.8<g<9.1 \mathrm{~m} / \mathrm{s}^{2}, 10.3<g<10.8 \mathrm{~m} / \mathrm{s}^{2} \quad-1.0 \mathrm{pts}$
outside the above ranges
-1.5 pts
- Error Calculation
1.0 pts


## Part 2a

- Measurements of H vs $\omega$

Calculation of $\omega$ using period measurements
At low speeds 10T is 0 K
At high speeds 20T is expected -0.2 pts
$\mathrm{H}-\omega$ table

- Plot of F vs $\omega$

Calculation of $\mathrm{F}=\mathrm{H}-\mathrm{h}_{0}$
Plot with axis labels
Drawing best fit line $\quad 0.5 \mathrm{pts}$
At least 6 different data in a wide range of $\omega \quad 0.6$ pts
No. of measurements 5: -0.2 pts
No of measurements 4:
No of measurements 3 or less:

- Calculations

Calculation of slope with unit
Dependence $F \alpha 1 / \omega^{2}$

## Part 2b

- Every correct item in the table


## Part 3

(At least 3 measurements at different orders are required)

- Wavelength measurement

Distance measurements and calculation of angle Calculation of $\lambda$

Credits to be subtracted from the total credit of 1.2:
If $\lambda$ is outside the range $600-700 \mathrm{~nm}$
If less then 3 measurements

- Measurement of $n$

Distance measurements and calculation of angle Realizing $\lambda / n$
Calculation of $n$
credits to be subtracted from the total credit of 2.3:
If $n$ is outside range 1.3-1.6
If less then 3 measurements
2.4 pts
0.8 pts
-0.4 pts
-0.6 pts
2.5 pts
1.0 pts
1.5 pts
3.5 pts
0.25 pts
5.5 pts
0.6 pts
0.4 pts
0.2 pts
0.5 pts
0.5 pts
3.5 pts
1.2 pts
0.6 pts
0.6 pts
-0.4 pts
-0.4 pts
2.3 pts
0.6 pts
0.8 pts
0.9 pts
-0.4 pts
-0.4 pts

## I. Ground-Penetrating Radar

Ground-penetrating radar (GPR) is used to detect and locate underground objects near the surface by means of transmitting electromagnetic waves into the ground and receiving the waves reflected from those objects. The antenna and the detector are directly on the ground and they are located at the same point.

A linearly polarized electromagnetic plane wave of angular frequency $\omega$ propagating in the z direction is represented by the following expression for its field:

$$
\begin{equation*}
E=E_{0} e^{-\alpha z} \cos (\omega t-\beta z), \tag{1}
\end{equation*}
$$

where $E_{o}$ is constant, $\alpha$ is the attenuation coefficient and $\beta$ is the wave number expressed respectively as follows
$\alpha=\omega\left\{\frac{\mu \varepsilon}{2}\left[\left(1+\frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}\right)^{1 / 2}-1\right]\right\}^{1 / 2}, \quad \beta=\omega\left\{\frac{\mu \varepsilon}{2}\left[\left(1+\frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}\right)^{1 / 2}+1\right]\right\}^{1 / 2}$
with $\mu, \varepsilon$, and $\sigma$ denoting the magnetic permeability, the electrical permittivity, and the electrical conductivity respectively.

The signal becomes undetected when the amplitude of the radar signal arriving at the object drops below 1/e ( $\approx 37 \%$ ) of its initial value. An electromagnetic wave of variable frequency ( $10 \mathrm{MHz}-1000 \mathrm{MHz}$ ) is usually used to allow adjustment of range and resolution of detection.

The performance of GPR depends on its resolution. The resolution is given by the minimum separation between the two adjacent reflectors to be detected. The minimum separation should give rise to a minimum phase difference of $180^{\circ}$ between the two reflected waves at the detector.

## Questions:

(Given : $\mu_{0}=4 \pi \times 10^{-7} \mathrm{H} / \mathrm{m}$ and $\varepsilon_{0}=8.85 \times 10^{-12} \mathrm{~F} / \mathrm{m}$ )

1. Assume that the ground is non-magnetic $\left(\mu=\mu_{0}\right)$ satisfying the condition $\left(\frac{\sigma}{\omega \varepsilon}\right)^{2}\langle\langle 1$. Derive the expression of propagation speed $v$ in terms of $\mu$ and $\varepsilon$, using equations (1) and (2) [1.0 pts].
2. Determine the maximum depth of detection of an object in the ground with conductivity of $1.0 \mathrm{mS} / \mathrm{m}$ and permittivity of $9 \varepsilon_{0}$, satisfying the condition $\left(\frac{\sigma}{\omega \varepsilon}\right)^{2}\left\langle\left\langle 1,\left(\mathrm{~S}=\mathrm{ohm}^{-1}\right.\right.\right.$; use $\left.\mu=\mu_{0}\right)$. [2.0 pts]
3. Consider two parallel conducting rods buried horizontally in the ground. The rods are 4 meter deep. The ground is known to have conductivity of $1.0 \mathrm{mS} / \mathrm{m}$ and permittivity of $9 \varepsilon_{0}$. Suppose the GPR measurement is carried out at a position aproximately above one of the rod. Assume point detector is used. Determine the minimum frequency required to get a lateral resolution of 50 cm [ 3.5 pts ].
4. To determine the depth of a buried rod $d$ in the same ground, consider the measurements carried out along a line perpendicular to the rod. The result is described by the following figure:


Graph of traveltime $t$ vs detector position $x, t_{\text {min }}=100 \mathrm{~ns}$.
Derive $t$ as a function of $x$ and determine $d[3.5 \mathrm{pts}]$.

## II. Sensing Electrical Signals

Some seawater animals have the ability to detect other creatures at some distance away due to electric currents produced by the creatures during the breathing processes or other processes involving muscular contraction. Some predators use this electrical signal to locate their preys, even when buried under the sands.

The physical mechanism underlying the current generation at the prey and its detection by the predator can be modeled as described by Figure II-1. The current generated by the prey flows between two spheres with positive and negative potential in the prey's body. The distance between the centers of the two spheres is $l_{s}$, each having a radius of $r_{s}$, which is much smaller than $l_{s}$. The seawater resistivity is $\rho$. Assume that the resistivity of the prey's body is the same as that of the surrounding seawater, implying that the boundary surrounding the prey in the figure can be ignored.


Figure II-1. A model describing the detection of electric power coming from a prey by its predator.

In order to describe the detection of electric power by the predator coming from the prey, the detector is modeled similarly by two spheres on the predator's body and in contact with the surrounding seawater, lying parallel to the pair in the prey's body. They are separated by a distance of $l_{d}$, each having a radius of $r_{d}$ which is much smaller than $l_{d}$. In this case, the center of the detector is located at a distance $y$ right above the source and the line connecting the two spheres is parallel to the electric field as shown in Figure II-1. Both $l_{s}$ and $l_{d}$ are also much smaller than $y$. The electric field strength along the line connecting the two spheres is assumed to be constant. Therefore the detector forms a closed circuit system connecting the prey, the surrounding seawater and the predator as described in Figure II-2.


Figure II-2. The equivalent closed circuit system involving the sensing predator, the prey and the surrounding seawater.

In the figure, $V$ is the voltage difference between the detector's spheres due to the electric field induced by the prey, $R_{m}$ is the inner resistance due to the surrounding sea water. Further, $V_{d}$ and $R_{d}$ are respectively the voltage difference between the detecting spheres and the resistance of the detecting element within the predator.

## Questions:

1. Determine the current density vector $\vec{j}$ (current per unit area) caused by a point current source $I_{s}$ at a distance $r$ in an infinite medium [1.5 pts]
2. Based on the law $\vec{E}=\rho \vec{j}$, determine the electric field strength $\vec{E}_{p}$ at the middle of the detecting spheres (at point P ) for a given current $I_{s}$ that flows between two spheres in the prey's body [2.0 pts].
3. Determine for the same current $I_{s}$, the voltage difference between the source spheres $\left(V_{s}\right)$ in the prey [1.5 pts]. Determine the resistance between the two source spheres $\left(R_{\mathrm{s}}\right)$ [0.5 pts] and the power produced by the source $\left(P_{s}\right)[0.5$ $p t s]$.
4. Determine $R_{m}[0.5 \mathrm{pts}], V_{d}[\mathbf{1 . 0} \mathbf{p t s}]$ in Figure II-2 and calculate also the power transferred from the source to the detector $\left(P_{d}\right)[0.5 \mathrm{pts}]$.
5. Determine the optimum value of $R_{d}$ leading to maximum detected power [1.5 $\boldsymbol{p t s}]$ and determine also the maximum power [0.5 pts].

## III. A Heavy Vehicle Moving on An Inclined Road



Figure III-1: A simplified model of a heavy vehicle moving on an inclined road.

The above figure is a simplified model of a heavy vehicle (road roller) with one rear and one front cylinder as its wheels on an inclined road with inclination angle of $\grave{e}$ as shown in Figure III-1. Each of the two cylinders has a total mass $\mathrm{M}\left(\mathrm{m}_{2}=\mathrm{m}_{3}=\mathrm{M}\right)$ and consists of a cylindrical shell of outer radius $R_{o}$, inner radius $R_{\mathrm{i}}=$ $0.8 R_{\mathrm{o}}$ and eight number of spokes with total mass 0.2 M . The mass of the undercarriage supporting the vehicle's body is negligible. The cylinder can be modeled as shown in Figure III-2. The vehicle is moving down the road under the influence of gravitational and frictional forces. The front and rear cylinder are positioned symmetrically with respect to the vehicle.


Figure III-2: A simplified model of the cylinders.

The static and kinetic friction coefficients between the cylinder and the road are $\mu_{s}$ and $\mu_{k}$ respectively. The body of the vehicle has a mass of 5 M , length of $L$ and thickness of $t$. The distance between the front and the rear cylinder is $2 l$ while the distance from the center of cylinder to the base of the vehicle's body is $h$. Assume that the rolling friction between the cylinder and its axis is negligible.

## Questions:

1. Calculate the moment of inertia of either cylinder [1.5 pts].
2. Draw all forces that act on the body, the front cylinder, and the rear one. Write down equations of motion for each part of them [2.5 pts].
3. The vehicle is assummed to move from rest, then freely move under gravitational influence. State all the possible types of motion of the system and derive their accelerations in terms of the given physical quantities [4.0 pts].
4. Assume that after the vehicle travels a distance $d$ by pure rolling from rest the vehicle enters a section of the road with all the friction coefficients drop to smaller constant values $\mu_{\mathrm{s}}$ ' and $\mu_{\mathrm{k}}$ ' such that the two cylinders start to slide. Calculate the linear and angular velocities of each cylinder after the vehicle has traveled a total distance of $s$ meters. Here we assume that d and s is much larger than the dimension of vehicle [2.0 pts]

## I. Determination of $e / k_{B}$ Through Electrolysis Process

## Background Theory

The electrolysis of water is described by the reaction :

$$
\begin{aligned}
& \mathrm{H}_{2} \mathrm{O} \rightarrow 2 \mathrm{H}^{+}+\mathrm{O}^{-2} \\
& 2 \mathrm{H}^{+}+2 \mathrm{e}^{-} \rightarrow \mathrm{H}_{2} ; \mathrm{O}^{-2} \rightarrow \frac{1}{2} \mathrm{O}_{2}+2 \mathrm{e}^{-}
\end{aligned}
$$

The reaction takes place when an electric current is supplied through a pair of electrodes immersed in the water. Assume that both gases produced in the reaction are ideal.

One of the gases produced by the reaction is kept in a test tube marked by arbitrary scale. By knowing the total charge transferred and the volume of the gas in the test tube the quantity $\boldsymbol{e} \boldsymbol{k}_{\boldsymbol{B}}$ can be determined, where $\mathbf{e}$ is the charge of electron and $\boldsymbol{k}_{\boldsymbol{B}}$ is the Boltzmann constant.

For the purpose mentioned above, this experiment is divided into two parts.
Part A: Calibration of the arbitrary scale on the test tube by using a dynamic method. This result will be used for part B

Part B: Determination of the physical quantity $\boldsymbol{e} \boldsymbol{k}_{\boldsymbol{B}}$ by means of water electrolysis
You are not obligedto carry out the two experiments ( part A and part B ) in alphabetical order

## The following physical quantities are assumed:

- Acceleration of gravity, $g=(9.78 \pm 0.01) \mathrm{ms}^{-2}$
- Ratio of internal vs external diameters of the test tube, $\alpha=0.82 \pm 0.01$

The local values of temperature $T$ and pressure $P$ will be provided by the organizer.

## List of tools and apparatus given for experiment (part A \& B):

- Insulated copper wires of three different diameters:

1. Brown of larger diameter
2. Brown of smaller diameter
3. Blue

- A regulated voltage source ( $0-60 \mathrm{~V}$, max.1A)
- A plastic container and a bottle of water.
- A block of brass with plastic clamp to keep the electrode in place without damaging the insulation of the wire.
- A digital stopwatch.
- A multimeter (beware of its proper procedure).
- A wooden test tube holder designed to hold the tube vertically.
- A multipurpose pipette
- A vertical stand.
- A bottle of white correction fluid for marking.
- A cutter
- A pair of scissors
- A roll of cellotape
- A steel ball
- A piece of stainless steel plate to be used as electrode.
- A test tube with scales.
- Graph papers.

Note that all scales marked on the graph papers and the apparatus for the experiments (e.g. the test tube) are of the same scale unit, but not calibrated in millimeter.

## EXPERIMENT

## Part A: Calibration of the arbitrary scale on the test tube

- Determine a dynamic method capable of translating the arbitrary length scale to a known scale available.
- Write down an expression that relates the measurable quantities from your experiment in terms of the scale printed on the test tube, and sketch the experiment set up.
- Collect and analyze the data from your experiment for the determination and calibration of the unknown length scale.


## Part B: Determination of physical quantity $\boldsymbol{e} / \boldsymbol{k}_{\boldsymbol{B}}$

- Set up the electrolysis experiment with a proper arrangement of the test tube in order to trap one of the gases produced during the reaction.
- Derive an equation relating the quantities: time $t$, current I , and water level difference $\Delta h$, measured in the experiment.
- Collect and analyze the data from your experiment. For simplicity, you may assume that the gas pressure inside the tube remains constant throughout the experiment.
- Determine the value of $\boldsymbol{e} / \mathbf{k}_{\boldsymbol{B}}$.

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| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |

## ANSWER FORM

## PART A

1. State the method of your choice and sketch the experimental set up of the method: [ 0.5 pts ]
2. Write down the expression relating the measurable quantities in your chosen method: [ 0.5 pts]. State all the approximations used in obtaining this expression [1.0 pts].
3. Collect and organize the data in the following orders: physical quantities, values, units [1.0 pts]
4. Indicate the quality of the calibration by showing the plot relating two independently measured quantities and mark the range of validity. [0.5 $p t s]$
5. Determine the smallest unit of the arbitrary scale in term of mm and its estimated error induced in the measurements. [1.5 pts]

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## PARTB

1. Sketch of the experimental set up. [1.0 pts]
2. Derive the following expression:

$$
I \Delta t=\frac{\mathrm{e}}{\mathrm{k}_{\mathrm{B}}} \frac{2 P\left(\pi r^{2}\right)}{T} \Delta \mathrm{~h} \quad[1.5 p t s]
$$

3. Collect and organize the data in the following format :physical quantities (value, units) [1.0 pts]
4. Determine the value of $e / k_{B}$ and its estimated error [1.5 pts]

## II. OPTICAL BLACK BOX

## Description

In this problem, the students have to identify the unknown optical components inside the cubic box. The box is sealed and has only two narrow openings protected by red plastic covering. The components should be identified by means of optical phenomena observed in the experiment. Ignore the small thickness effect of the plastic covering layer.

A line going through the centers of the slits is defined as the axis of the box. Apart from the red plastic coverings, there are three (might be identical or different) elements from the following list:

- Mirror, either plane or spherical
- Lens, either positive or negative
- Transparent plate having parallel flat surfaces (so called plane-parallel plate)
- Prism
- Diffraction grating.

The transparent components are made of material with a refractive index of 1.47 at the wavelength used.

## Apparatus available:

- A laser pointer with a wavelength of 670 nm . CAUTION: DO NOT LOOK DIRECTLY INTO THE LASER BEAM.
- An optical rail
- A platform for the cube, movable along the optical rail
- A screen which can be attached to the end of the rail, and detached from it for other measurements.
- A sheet of graph paper which can be pasted on the screen by cellotape.
- A vertical stand equipped with a universal clamp and a test tube with arbitrary scales, which are also used in the Problem I.

Note that all scales marked on the graph papers and the apparatus for the experiments are of the same scale unit, but not calibrated in millimeter.

## The Problem

Identify each of the three components and give its respective specification:

| Possible type of component | Specification required |
| :---: | :--- |
| mirror | radius of curvature, angle between the mirror axis and <br> the axis of the box |
| lens* | positive or negative, its focal length, and its position inside the <br> box |
| plane-parallel plate | thickness, the angle between the plate and the axis of the box |
| prism | apex angle, the angle between one of its deflecting sides and <br> the axis of the box |
| diffraction grating* | line spacing, direction of the lines, and its position inside the <br> box |

- implies that its plane is at right angle to the axis of the box

Express your final answers for the specification parameters of each component (e.g. focal length, radius of curvature) in terms of millimeter, micrometer or the scale of graph paper.

You don't have to determine the accuracy of the results.

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| :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |

## ANSWER FORM

1. Write down the types of the optical components inside the box :
no.1.
[0.5 pts]
no.2..
[0.5 pts]
no.3.
[0.5 pts]
2. The cross section of the box is given in the figure below. Add a sketch in the figure to show how the three components are positioned inside the box. In your sketch, denote each component with its code number in answer 1.
[0.5 pts for each correct position]


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|  |  |  |  |  |

3. Add detailed information with additional sketches regarding arrangement of the optical components in answer 2 , such as the angle, the distance of the component from the slit, and the orientation or direction of the components. [1.0 pts]

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| :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |

4. Summarize the observed data [ 0.5 pts ], determine the specification of the optical component no. 1 by deriving the appropriate formula with the help of drawing [1.0 pts], calculate the specifications in question and enter your answer in the box below [0.5 pts].

| Name of component no.1 | Specification |
| :--- | :--- |
|  |  |
|  |  |


| Country | Student No. | Experiment No. | Page No. | Total Pages |
| :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |

5. Summarize the observed data [ 0.5 pts ], determine the specification of the optical component no. 2 by deriving the appropriate formula with the help of drawing [1.0 pts], calculate the specifications in question and enter your answer in the box below [0.5 pts].

| Name of component no.2 | Specification |
| :---: | :---: |
|  |  |
|  |  |


| Country | Student No. | Experiment No. | Page No. | Total Pages |
| :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |

6. Summarize the observed data [ 0.5 pts ], determine the specification of the optical component no.3 by deriving the appropriate formula with the help of drawing [1.0 pts], calculate the specifications in question and enter your answer in the box below [0.5 pts].

| Name of component no.3 | Specification |
| :---: | :---: |
|  |  |
|  |  |

33rd
INTERNATIONAL
PHYSICS OLYMPIAD


## THEORETICAL COMPETITION

Tuesday, July $23^{\text {rd }}, 2002$

## Solution I: Ground-Penetrating Radar

1. Speed of radar signal in the material $v_{m}$ :

$$
\begin{align*}
& \omega t-\beta z=\text { constant } \rightarrow \beta \mathrm{z}=- \text { constant }+\omega \mathrm{t}(0.2 \mathrm{pts}) \\
& v_{m}=\frac{\omega}{\beta} \\
& v_{m}=\frac{1}{\omega\left\{\frac{\mu \varepsilon}{2}\left[\left(1+\frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}\right)^{1 / 2}+1\right]\right\}^{1 / 2}} \quad(0.4 \mathrm{pts})  \tag{0.4pts}\\
& v_{m}=\frac{1}{\left\{\frac{\mu \varepsilon}{2}(1+1)\right\}^{1 / 2}}=\frac{1}{\sqrt{\mu \varepsilon}} \tag{0.4pts}
\end{align*}
$$

2. The maximum depth of detection (skin depth, $\boldsymbol{\delta}$ ) of an object in the ground is inversely proportional to the attenuation constant:
(0.5 pts)
(0.3 pts)
(0.2 pts)

$$
\begin{aligned}
& \delta=\frac{1}{a}=\frac{1}{\omega\left\{\frac{\mu \varepsilon}{2}\left[\left(1+\frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}\right)^{1 / 2}-1\right]\right\}^{1 / 2}}=\frac{1}{\omega\left\{\frac{\mu \varepsilon}{2}\left[\left(1+\frac{1}{2} \frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}\right)-1\right]\right\}^{1 / 2}}=\frac{1}{\omega\left\{\frac{\mu \varepsilon}{2} \cdot \frac{1}{2} \frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}\right\}^{1 / 2}} \\
& \delta=\left(\frac{2}{\sigma}\right)\left(\frac{\varepsilon}{\mu}\right)^{1 / 2} .
\end{aligned}
$$

Numerically $\delta=\frac{\left(5.31 \sqrt{\varepsilon_{r}}\right)}{\sigma} \mathrm{m}$, where $\sigma$ is in $\mathrm{mS} / \mathrm{m}$. $\quad \mathbf{( 0 . 5 ~ p t s ) ~}$
For a medium with conductivity of $1.0 \mathrm{mS} / \mathrm{m}$ and relative permittivity of 9 , the skin depth

$$
\boldsymbol{\delta}=\frac{(5.31 \sqrt{9})}{1.0}=15.93 \mathrm{~m}
$$

3. Lateral resolution:

$$
\underbrace{\text { Antenna }}_{r \operatorname{rod}} r=\left(\frac{\lambda d}{2}+\frac{\lambda^{2}}{16}\right)^{1 / 2}
$$

## (1.0 pts)

$\mathrm{r}=0.5 \mathrm{~m}, \mathrm{~d}=4 \mathrm{~m}: \frac{1}{2}=\left(\frac{4 \lambda}{2}+\frac{\lambda^{2}}{16}\right)^{1 / 2}, \quad \lambda^{2}+32 \lambda-4=0$
The wavelength is $\lambda=0.125 \mathrm{~m}$.
( 0.3 pts ) $+(0.2 \mathrm{pts})$
The propagation speed of the signal in medium is

$$
\begin{align*}
& v_{m}=\frac{1}{\sqrt{\mu \varepsilon}}=\frac{1}{\sqrt{\mu_{o} \mu_{r} \varepsilon_{o} \varepsilon_{r}}}=\frac{1}{\sqrt{\mu_{o} \varepsilon_{o}}} \frac{1}{\sqrt{\mu_{r} \varepsilon_{r}}} \\
& v_{m}=\frac{c}{\sqrt{\mu_{r} \varepsilon_{r}}}=\frac{0.3}{\sqrt{\varepsilon_{r}}} \mathrm{~m} / \mathrm{ns}, \text { where } c=\frac{1}{\sqrt{\mu_{o} \varepsilon_{o}}} \text { and } \mu_{\mathrm{r}}=1 \\
& v_{m}=0.1 \mathrm{~m} / \mathrm{ns}=10^{8} \mathrm{~m} / \mathrm{s} \tag{0.5pts}
\end{align*}
$$

The minimum frequency need to distinguish the two rods as two separate objects is

$$
\left.\begin{array}{rl} 
& f_{\min }=\frac{v}{\lambda}  \tag{0.5pts}\\
f_{\min }= & \frac{0.3}{\sqrt{9}} \\
0.125
\end{array} 10^{9} \mathrm{~Hz}=800 \mathrm{MHz} \quad \mathbf{( 0 . 5 ~ \mathbf { ~ p t }} \quad \mathrm{(0.3} \mathbf{~ p t s}\right)+(\mathbf{0 . 2 0} \mathbf{~ p t s})
$$

4. Path of EM waves for some positions on the ground surface


The traveltime as function of $x$ is

$$
\begin{align*}
& \left(\frac{t v}{2}\right)^{2}=d^{2}+x^{2},  \tag{1.0pts}\\
& t(x)=\sqrt{\frac{4 d^{2}+4 x^{2}}{v}} \\
& t(x)=\frac{2 \sqrt{\varepsilon_{1 r}}}{0.3} \sqrt{d^{2}+x^{2}}
\end{align*}
$$

(1.0 pts)


For $x=0$

$$
100=2 \times(3 / 0.3) d
$$

$$
\begin{equation*}
d=5 \mathrm{~m} \tag{0.5pts}
\end{equation*}
$$

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# THEORETICAL COMPETITION 

Tuesday, July $23^{\text {rd }}, 2002$

## Solution II: Sensing Electrical Signals

1. When a point current source $I_{s}$ is in infinite isotropic medium, the current density vector at a distance $r$ from the point is
$\vec{j}=\frac{I_{s}}{4 \pi r^{3}} \vec{r}$
[+1.5 pts] (without vector notation, $-0.5 \mathrm{pts})$


Assuming that the resistivities of the prey body and that of the surrounding seawater are the same, implying the elimination of the boundary surrounding the prey, the two spheres seem to be in infinite isotropic medium with the resistivity of $\rho$. When a small sphere produces current at a rate $I_{s}$, the current flux density at a distance $r$ from the sphere's center is also
$\vec{j}=\frac{I_{s}}{4 \pi r^{3}} \vec{r}$

The seawater resistivity is $\rho$, therefore the field strength at $r$ is

$$
\vec{E}(\vec{r})=\rho \vec{j}=\frac{\rho I_{s}}{4 \pi r^{3}} \vec{r} \quad[+0.2 \mathrm{pts}]
$$

In the model, we have two small spheres. One is at positive voltage relative to the other therefore current $I_{s}$ flows from the positively charged sphere to the negatively charged sphere. They are separated by $l_{s}$. The field strength at $\mathrm{P}(0, \mathrm{y})$ is:

$$
\begin{aligned}
\vec{E}_{p} & =\vec{E}_{+}+\vec{E}_{-} \quad[+0.8 \mathrm{pts}] \\
& =\frac{\rho I_{s}}{4 \pi}\left[\frac{1}{\left(\left(\frac{l_{s}}{2}\right)^{2}+y^{2}\right)^{\frac{3}{2}}}\left(-\frac{l_{s}}{2} i+y j\right)+\frac{1}{\left(\left(\frac{l_{s}}{2}\right)^{2}+y^{2}\right)^{\frac{3}{2}}}\left(-\frac{l_{s}}{2} i-y j\right)\right] \\
& =\frac{\rho I_{s}}{4 \pi}\left[\frac{l_{s}(-i)}{\left(\left(\frac{l_{s}}{2}\right)^{2}+y^{2}\right)^{\frac{3}{2}}}\right] \\
\vec{E}_{p} & \approx \frac{\rho I_{s} l_{s}}{4 \pi y^{3}}(-i) \quad \text { for } \mathrm{ls} \ll \mathrm{y} \quad[+1.0 \mathrm{pts}]
\end{aligned}
$$

3. The field strength along the axis between the two source spheres is:

$$
\vec{E}(x)=\frac{\rho I_{s}}{4 \pi}\left(\frac{1}{\left(x-\frac{l_{s}}{2}\right)^{2}}+\frac{1}{\left(x+\frac{l_{s}}{2}\right)^{2}}\right)(-i) \quad[+0.5 \mathrm{pts}]
$$

The voltage difference to produce the given current $I_{s}$ is

$$
\begin{aligned}
V_{s} & =\Delta V=V_{+}-V_{-}=-\int_{\left(-\frac{l_{s}}{2}+r_{s}\right)}^{\left(\frac{l_{s}}{2}-r_{s}\right)} \vec{E}(x) \cdot d \vec{x}=-\frac{\rho I_{s}}{4 \pi} \int\left(\frac{1}{\left(x-\frac{l_{s}}{2}\right)^{2}}+\frac{1}{\left(x+\frac{l_{s}}{2}\right)^{2}}\right)(-i) .(i d x) \quad[+0.5 \mathrm{pts}] \\
& \left.=\frac{\rho I_{s}}{4 \pi}\left[\frac{1}{-2+1}\left(\frac{1}{\left(\frac{l_{s}}{2}-r_{s}-\frac{l_{s}}{2}\right)}-\frac{1}{\left(-\frac{l_{s}}{2}+r_{s}-\frac{l_{s}}{2}\right)}\right)+\frac{1}{-2+1} \frac{1}{\left(\frac{l_{s}}{2}-r_{s}+\frac{l_{s}}{2}\right)}-\frac{1}{\left(-\frac{l_{s}}{2}+r_{s}+\frac{l_{s}}{2}\right)}\right)\right] \\
& =\frac{\rho I_{s}}{4 \pi}\left(\frac{2}{r_{s}}-\frac{2}{l_{s}-r_{s}}\right)=\frac{2 \rho I_{s}}{4 \pi}\left(\frac{l_{s}-r_{s}-r_{s}}{\left(l_{s}-r_{s}\right) r_{s}}\right)=\frac{\rho I_{s}}{2 \pi r_{s}}\left(\frac{l_{s}-2 r_{s}}{l_{s}-r_{s}}\right) \\
V_{s} & =\Delta V \approx \frac{\rho I_{s}}{2 \pi r_{s}} \quad \text { for } l_{s} \gg r_{s} . \quad[+0.5 \mathrm{pts}]
\end{aligned}
$$

The resistance between the two source spheres is:

$$
R_{s}=\frac{V_{s}}{I_{s}}=\frac{\rho}{2 \pi r_{s}}
$$

[ +0.5 pts ]
The power produced by the source is:

$$
P=I_{s} V_{s}=\frac{\rho I_{s}{ }^{2}}{2 \pi r_{s}}
$$

[+0.5 pts]
4.

$V$ is the voltage difference between the detector's spheres due to the electric field induced bythe prey, $R_{m}$ is the inner resistance due to the surrounding sea water. $V_{d}$ and $R_{d}$ are respectively the voltage difference between the detecting spheres and the resistance of the detecting element within the predator and $i_{d}$ is the current flowing in the closed circuit.

Analog to the resistance between the two source spheres, the resistance of the medium with resistivity $\rho$ between the detector spheres, each having a radius of $r_{d}$ is:
$R_{m}=\frac{\rho}{2 \pi r_{d}}$
[+0.5 pts]
Since $l_{d}$ is much smaller than $y$, the electric field strength between the detector spheres can be assumed to be constant, that is:

$$
E=\frac{\rho I_{s} l_{s}}{4 \pi y^{3}} \quad[+0.2 \mathrm{pts}]
$$

Therefore, the voltage difference present in the medium between the detector spheres is:
$V=E l_{d}=\frac{\rho I_{s} l_{s} l_{d}}{4 \pi y^{3}} \quad[+0.3 \mathrm{pts}]$

The voltage difference across the detector spheres is:
$V_{d}=V \frac{R_{d}}{R_{d}+R_{m}}=\frac{\rho I_{s} l_{s} l_{d}}{4 \pi y^{3}} \frac{R_{d}}{R_{d}+\frac{\rho}{2 \pi r_{d}}}$
[ +0.5 pts ]
The power transferred from the source to the detector is:
$P_{d}=i_{d} V_{d}=\frac{V}{R_{d}+R_{m}} V_{d}=\left(\frac{\rho I_{s} l_{s} l_{d}}{4 \pi y^{3}}\right)^{2} \frac{R_{d}}{\left(R_{d}+\frac{\rho}{2 \pi r_{d}}\right)^{2}}$
[ +0.5 pts ]
5. $\quad P_{d}$ is maximum when

$$
R_{t}=\frac{R_{d}}{\left(R_{d}+\frac{\rho}{2 \pi r_{d}}\right)^{2}}=\frac{R_{d}}{\left(R_{d}+R_{m}\right)^{2}} \quad \text { is maximum } \quad[+0.5 \mathrm{pts}]
$$

Therefore,

$$
\begin{aligned}
& \frac{d R_{t}}{d R_{d}}=\frac{1\left(R_{d}+R_{m}\right)^{2}-R_{d} 2\left(R_{d}+R_{m}\right)}{\left(R_{d}+R_{m}\right)^{4}}=0 \quad[+0.5 \mathrm{pts}] \\
& \left(R_{d}+R_{m}\right)-2 R_{d}=0 \\
& R_{d}^{\text {optimum }}=R_{m}=\frac{\rho}{2 \pi r_{d}} \quad[+0.5 \mathrm{pts}]
\end{aligned}
$$

The maximum power is:

$$
P_{d}^{\max i m u m}=\left(\frac{\rho I_{s} l_{s} l_{d}}{4 \pi y^{3}}\right)^{2} \frac{\pi r_{d}}{2 \rho}=\frac{\rho\left(I_{s} l_{s} l_{d}\right)^{2} r_{d}}{32 \pi y^{6}}
$$

[ +0.5 pts ]

## solution t3 : . A Heavy Vehicle Moving on An Inclined Road



To simplify the model we use the above figure with $h_{1}=h+0.5 t$ $\mathrm{R}_{\mathrm{o}}=\mathrm{R}$

1. Calculation of the moment inertia of the cylinder
$\mathrm{R}_{\mathrm{i}}=0.8 \mathrm{R}_{0}$
Mass of cylinder part : $\mathrm{m}_{\text {cylinder }}=0.8 \mathrm{M}$
Mass of each rod $: \mathrm{m}_{\mathrm{rod}}=0.025 \mathrm{M}$


$$
\begin{array}{rlrl}
I=\oint_{\text {wholepart }} r^{2} d m=\oint_{\text {cyl.shell }} r^{2} d m+\oint_{\text {rod1 }} r^{2} d m+\ldots+\oint_{\text {rod }} r^{2} d m & 0.4 \mathrm{pts} \\
\oint_{\text {cyl.shell }} r^{2} d m & =2 \pi \sigma \int_{R i}^{R o} r^{3} d r=0.5 \pi \sigma\left(R_{o}^{4}-R_{i}^{4}\right)=0.5 m_{\text {cylinder }}\left(R_{o}^{2}+R_{i}^{2}\right) & \\
& =0.5(0.8 M) R^{2}(1+0.64)=0.656 M R^{2} & 0.5 \mathrm{pts} \\
\oint_{\text {rod }} r^{2} d m & =\lambda \int_{0}^{R i n} r^{2} d r=\frac{1}{3} \lambda R_{\text {in }}^{3}=\frac{1}{3} m_{\text {rod }} R_{i n}^{2}=\frac{1}{3} 0.025 M\left(0.64 R^{2}\right)=0.00533 M R^{2} & 0.5 \mathrm{pts}
\end{array}
$$

The moment inertia of each wheel becomes

$$
I=0.656 M R^{2}+8 x 0.00533 M R^{2}=0.7 M R^{2}
$$

## 2. Force diagram and balance equations:

To simplify the analysis we devide the system into three parts: frame (part1) which mainly can be treated as flat homogeneous plate, rear cylinders (two cylinders are treated collectively as part 2 of the system), and front cylinders (two front cylinders are treated collectively as part 3 of the system).

Part 1 : Frame


The balance equation related to the forces work to this parts are:

Required conditions:
Balance of force in the horizontal axis
$m_{1} g \sin \theta-f_{12 h}-f_{13 h}=m_{1} a$
(1) 0.2 pts

Balance of force in the vertical axis
$m_{1} g \cos \boldsymbol{\theta}=N_{12}+N_{13}$
(2) 0.2 pts

Then torsi on against O is zero, so that
$\mathrm{N}_{12} l-\mathrm{N}_{13} l+f_{12 h} h_{1}+f_{13 h} h_{1}=0$
(3) 0.2 pts

Part two : Rear cylinder

0.25 pts

From balance condition in rear wheel :
$\mathrm{f}_{21 \mathrm{~h}}-f_{2}+M g \sin \boldsymbol{\theta}=M a$
$\mathrm{N}_{2}-\mathrm{N}_{21}-M g \cos \boldsymbol{\theta}=0$
For pure rolling:

$$
\begin{align*}
& f_{2} R=I \boldsymbol{\alpha}_{2}=I \frac{a_{2}}{R} \\
& \text { or } \mathrm{f}_{2}=\frac{I}{R^{2}} \mathrm{a} \tag{6}
\end{align*}
$$

(5) 0.15 pts

For rolling with sliding:

$$
\begin{equation*}
\mathrm{F}_{2}=\mathrm{u}_{\mathrm{k}} \mathrm{~N}_{2} \tag{7}
\end{equation*}
$$

$$
0.2 \text { pts }
$$

## Part Three : Front Cylinder:


0.25 pts

From balance condition in the front whee l:
$\mathrm{f}_{31 \mathrm{~h}}-f_{3}+M g \sin \theta=M a$
(8) 0.15 pts
$\mathrm{N}_{3}-N_{31}-M g \cos \boldsymbol{\theta}=0$
(9) 0.15 pts

For pure rolling:
$f_{3} R=I \alpha_{3}=I \frac{a_{3}}{R}$
or $\mathrm{f}_{3}=\frac{I}{R^{2}} \mathrm{a}$
For rolling with sliding:

$$
\begin{equation*}
\mathrm{F}_{3}=\mathrm{u}_{\mathrm{k}} \mathrm{~N}_{3} \tag{11}
\end{equation*}
$$

## 3. From equation (2), (5) and (9) we get

$$
\begin{array}{r}
m_{1} g \cos \theta=N_{2}-m_{2} g \cos \theta+N_{3}-m_{3} g \cos \theta \\
N_{2}+N_{3}=\left(m_{1}+m_{2}+m_{3}\right) g \cos \theta=7 M g \cos \theta \tag{12}
\end{array}
$$

And from equation (3), (5) and (8) we get
$\left(\mathrm{N}_{3}-\mathrm{Mg} \cos \theta\right) \mathrm{l}-\left(\mathrm{N}_{2}-\mathrm{Mg} \cos \theta\right) \mathrm{l}=\mathrm{h}_{1}\left(\mathrm{f}_{2}+\mathrm{Ma}-\mathrm{Mg} \sin \theta+\mathrm{f}_{3}+\mathrm{Ma}-\mathrm{Mg} \sin \theta\right)$
$\left(\mathrm{N}_{3}-\mathrm{N}_{2}\right)=\mathrm{h}_{1}\left(\mathrm{f}_{2}+2 \mathrm{Ma}-2 \mathrm{Mg} \sin \theta+\mathrm{f}_{3}\right) / \mathrm{l}$

## Equations 12 and 13 are given $\mathbf{0 . 2 5} \mathbf{p t s}$

## CASE ALL CYLINDER IN PURE ROLLING

From equation (4) and (6) we get

$$
\begin{equation*}
\mathrm{f}_{21 \mathrm{~h}}=\left(\mathrm{I} / \mathrm{R}^{2}\right) a+M a-M g \sin \theta \tag{14}
\end{equation*}
$$

From equation (8) and (10) we get

$$
\begin{equation*}
\mathrm{f}_{31 \mathrm{~h}}=\left(\mathrm{I} / \mathrm{R}^{2}\right) \mathrm{a}+\mathrm{Ma}-\mathrm{Mg} \sin \theta \tag{15}
\end{equation*}
$$

Then from eq. (1) , (14) and (15) we get

$$
5 M g \sin \theta-\left\{\left(I / R^{2}\right) a+M a-M g \sin \theta\right\}-\left\{\left(I / R^{2}\right) a+M a-M g \sin \theta\right\}=m_{1} a
$$

$$
7 \mathrm{Mg} \sin \theta=\left(2 \mathrm{I} / \mathrm{R}^{2}+7 \mathrm{M}\right) \mathrm{a}
$$

$$
a=\frac{7 M g \sin \theta}{7 M+2 \frac{I}{R^{2}}}=\frac{7 M g \sin \theta}{7 M+2 \frac{0.7 M R^{2}}{R^{2}}}=0.833 g \sin \theta
$$

$$
N_{3}=\frac{7 M}{2} g \cos \theta+\frac{h_{1}}{l}\left[\left(M+\frac{I}{R^{2}}\right) \times 0.833 g \sin \theta-M g \sin \theta\right]
$$

$$
=3.5 M g \cos \theta+\frac{h_{1}}{l}[(M+0.7 M) \times 0.833 g \sin \theta-M g \sin \theta]
$$

$$
=3.5 \mathrm{Mg} \cos \theta+0.41 \frac{h_{1}}{l} M g \sin \theta
$$

$$
\begin{aligned}
N_{2} & =\frac{7 M}{2} g \cos \theta-\frac{h_{1}}{l}\left[\left(\frac{I}{R^{2}}+M\right) \times 0.833 g \sin \theta-M g \sin \theta\right] \\
& =3.5 \mathrm{~g} \cos \theta-\frac{h_{1}}{l}\left[(0.7 M+M) \frac{7 M g \sin \theta}{0.7 M+7 M}-2 M g \sin \theta\right] \\
& =3.5 \mathrm{~g} \cos \theta-0.41 \frac{h_{1}}{l} M g \sin \theta
\end{aligned}
$$

$$
0.2 \text { pts }
$$

The Conditions for pure rolling:

$$
\begin{array}{ll}
f_{2} \leq \mu_{s} N_{2} & \text { and } f_{3} \leq \mu_{s} N_{3} \\
\frac{\mathrm{I}_{2}}{\mathrm{R}_{2}^{2}} \mathrm{a} \leq \mu_{s} N_{2} & \text { and } \frac{\mathrm{I}_{3}}{\mathrm{R}_{3}^{2}} \mathrm{a} \leq \mu_{s} N_{3}
\end{array}
$$

The left equation becomes
$0.7 M \times 0.833 g \sin \theta \leq \mu_{s}\left(3.5 \mathrm{Mg} \cos \theta-0.41 \frac{h_{1}}{l} M g \sin \theta\right)$
$\tan \theta \leq \frac{3.5 \mu_{s}}{0.5831+0.41 \mu_{s} \frac{h_{1}}{l}}$

While the right equation becomes
$0.7 m \times 0.833 g \sin \theta \leq \mu_{s}\left(3.5 m g \cos \theta+0.41 \frac{h_{1}}{l} m g \sin \theta\right)$
$\tan \theta \leq \frac{3.5 \mu_{\mathrm{s}}}{0.5831-0.41 \mu_{\mathrm{s}} \frac{h_{1}}{l}}$
0.1 pts

## CASE ALL CYLINDER SLIDING

From eq. (4) $\mathrm{f}_{21 \mathrm{~h}}=\mathrm{Ma}+\mathrm{u}_{\mathrm{k}} \mathrm{N}_{2}-\mathrm{Mg} \sin \theta$
From eq. (8) $f_{31 \mathrm{~h}}=M a+u_{k} N_{3}-M g \sin \theta$
0.15 pts

From eq. (18) and 19 :
$5 M g \sin \theta-\left(M a+u_{k} N_{2}-M g \sin \theta\right)-\left(M a+u_{k} N_{3}-M g \sin \theta\right)=m_{1} a$

$$
a=\frac{7 M g \sin \theta-\mu_{k} N_{2}-\mu_{k} N_{3}}{7 M}=g \sin \theta-\frac{\mu_{k}\left(N_{2}+N_{3}\right)}{7 M}
$$

0.15 pts
0.2 pts

$$
N_{3}+N_{2}=7 M g \cos \theta
$$

From the above two equations we get :

$$
\mathrm{a}=g \sin \theta-\mu_{k} g \cos \theta
$$

The Conditions for complete sliding: are the opposite of that of pure rolling

$$
\begin{array}{ll}
f_{2}>\mu_{s} N_{2}^{\prime} & \text { and } f_{3}>\mu_{s} N_{3}^{\prime} \\
\left.\frac{\mathrm{I}_{2}}{\mathrm{R}_{2}^{2}} \mathrm{a}>\boldsymbol{\mu}_{s} N_{2}^{\prime} \quad \text { and } \frac{\mathrm{I}_{3}}{\mathrm{R}_{3}^{2}} \mathrm{a}\right\rangle \mu_{s} N_{3}^{\prime}
\end{array}
$$

Where $\mathrm{N}_{2}{ }^{\prime}$ and $\mathrm{N}_{3}$ ' is calculated in case all cylinder in pure rolling.
0.1 pts

Finally weget
$\tan \theta>\frac{3.5 \mu_{s}}{0.5831+0.41 \mu_{s} \frac{h_{1}}{l}} \quad$ and $\left.\quad \tan \theta\right\rangle \frac{3.5 \mu_{s}}{0.5831-0.41 \mu_{s} \frac{h_{1}}{l}} \quad 0.2 \mathrm{pts}$
The left inequality finally become decisive.

## CASE ONE CYLINDER IN PURE ROLLING AND ANOTHER IN SLIDING CONDITION

\{ For example $\mathrm{R}_{3}$ (front cylinders) pure rolling while $\mathrm{R}_{2}$ (Rear cylinders) sliding\}

From equation (4) we get

$$
\begin{equation*}
F_{21 h}=m_{2} a+u_{k} N_{2}-m_{2} g \sin \theta \tag{22}
\end{equation*}
$$

0.15 pts

From equation (5) we get

$$
\begin{equation*}
f_{31 h}=m_{3} a+\left(I / R^{2}\right) a-m_{3} g \sin \theta \tag{23}
\end{equation*}
$$

0.15 pts

Then from eq. (1) , (22) and (23) we get
$m_{1} g \sin \theta-\left\{m_{2} a+u_{k} N_{2}-m_{2} g \sin \theta\right\}-\left\{m_{3} a+\left(I / R^{2}\right) a-m_{3} g \sin \theta\right\}=m_{1} a$
$m_{1} g \sin \theta+m_{2} g \sin \theta+m_{3} \sin \theta-u_{k} N_{2}=\left(I / R^{2}+m_{3}\right) a+m_{2} a+m_{1} a$
$5 M g \sin \theta+M g \sin \theta+M g \sin \theta-u_{k} N_{2}=(0.7 M+M) a+M a+5 M a$
$a=\frac{7 M g \sin \theta-\mu_{\mathrm{k}} \mathrm{N}_{2}}{7.7 M}=0.9091 g \sin \theta-\frac{\mu_{\mathrm{k}} \mathrm{N}_{2}}{7.7 M}$
0.2 pts
$N_{3}-N_{2}=\frac{h_{1}}{l}\left(\mu_{k} N_{2}+\frac{I}{R^{2}} a+2 M a-2 M g \sin \theta\right)$
$N_{3}-N_{2}=\frac{h_{1}}{l}\left(\mu_{k} N_{2}+2.7 M \times 0.9091 g \sin \theta-2.7 \mu_{k} N_{2} / 7.7-2 M g \sin \theta\right)$
$N_{3}-N_{2}\left(1+0.65 \mu_{k} \frac{h_{1}}{l}\right)=0.4546 M g \sin \theta$
$N_{3}+N_{2}=7 M g \cos \theta$
Therefore we get

$$
\begin{align*}
& N_{2}=\frac{7 M g \cos \theta-0.4546 M g \sin \theta}{2+0.65 \mu_{k} \frac{h_{1}}{l}}  \tag{25}\\
& N_{3}=7 M g \cos \theta-\frac{7 M g \cos \theta-0.4546 M g \sin \theta}{2+0.65 \mu_{k} \frac{h_{1}}{l}}
\end{align*}
$$

Then we can substitute the results above into equation (16) to get the following result $a=0.9091 g \sin \theta-\frac{\mu_{\mathrm{k}} \mathrm{N}_{2}}{7.7 M}=0.9091 g \sin \theta-\frac{\mu_{\mathrm{k}}}{7.7} \frac{7 g \cos \theta-0.4546 g \sin \theta}{2+0.65 \mu_{k} \frac{h_{1}}{l}}$

$$
0.2 \text { pts }
$$

The Conditions for this partial sliding is:

$$
\begin{array}{ll}
f_{2} \leq \mu_{s} N_{2}^{\prime} & \text { and } f_{3}>\mu_{s} N_{3}^{\prime} \\
\frac{\mathrm{I}}{\mathrm{R}^{2}} \mathrm{a} \leq \mu_{s} N_{2}^{\prime} & \text { and } \left.\frac{\mathrm{I}}{\mathrm{R}^{2}} \mathrm{a}\right\rangle \mu_{s} N_{3}^{\prime} \tag{27}
\end{array}
$$

where $N_{2}^{\prime}$ and $N_{3}^{\prime}$ are normal forces for pure rolling condition
4. Assumed that after rolling d meter all cylinder start to sliding until reaching the end of incline road (total distant is s meter). Assummed that $\eta$ meter is reached in $\mathrm{t}_{1}$ second.
$v_{t 1}=v_{o}+a t_{1}=0+a_{1} t_{1}=a_{1} t_{1}$
$d=v_{o} t_{1}+\frac{1}{2} a_{1} t_{1}^{2}=\frac{1}{2} a_{1} t_{1}^{2}$
$t_{1}=\sqrt{\frac{2 d}{a_{1}}}$
0.5 pts
$v_{\mathrm{t} 1}=a_{1} \sqrt{\frac{2 d}{a_{1}}}=\sqrt{2 d a_{1}}=\sqrt{2 d 0.833 g \sin \theta}=\sqrt{1.666 d g \sin \boldsymbol{\theta}}$
The angular velocity after rolling $d$ meters is same for front and rear cylinders:
$\omega_{t 1}=\frac{v_{t 1}}{R}=\frac{1}{R} \sqrt{1.666 d g \sin \theta}$

## 0.5 pts

Then the vehicle sliding untill the end of declining road. Assumed that the time needed by vehicle to move from d position to the end of the declining road is $\mathrm{t}_{2}$ second.
$v_{t 2}=v_{t 1}+a_{2} t_{2}=\sqrt{1.666 \operatorname{dg} \sin \boldsymbol{\theta}}+a_{2} t_{2}$
$s-d=v_{t 1} t_{2}+\frac{1}{2} a_{2} t_{2}^{2}$
$t_{2}=\frac{-v_{t 1}+\sqrt{v_{t 1}^{2}+2 a_{2}(s-d)}}{a_{2}}$
0.4 pts
$v_{t 2}=\sqrt{1.666 d g \sin \boldsymbol{\theta}}-v_{t 1}+\sqrt{v_{t 1}^{2}+2 a_{2}(s-d)}$
Inserting $\mathrm{v}_{\mathrm{t} 1}$ and $\mathrm{a}_{2}$ from the previous results we get the final results.
For the angular velocity, while sliding they receive torsion:

$$
\begin{align*}
& \tau=\mu_{k} N R \\
& \alpha=\frac{\tau}{I}=\frac{\mu_{k} N R}{I}  \tag{31}\\
& \omega_{t 2}=\omega_{t 1}+\alpha t_{2}=\frac{1}{R} \sqrt{1.666 d g \sin \theta}+\frac{\mu_{k} N R}{I} \frac{-v_{t 1}+\sqrt{v_{t 1}^{2}+2 a_{2}(s-d)}}{a_{2}}
\end{align*}
$$

## SOLUTION EXPERIMENT I

## PART A

## 1. [Total 0.5 pts$]$

The experimental method chosen for the calibration of the arbitrary scale is a simple pendulum method [0.3 pts]


Figure 1. Sketch of the experimental set up [0.2 pts]

## 2. [Total 1.5 pts$]$

The expression relating the measurable quantities: [0.5 pts]

$$
T_{\text {osc }}=2 \pi \sqrt{\frac{l}{g}} ; T_{\text {osc }}{ }^{2}=4 \pi^{2} \frac{l}{g}
$$

Approximations:

$$
\sin \theta \approx \theta \quad[0.5 \mathrm{pts}]
$$

mathematical pendulum (mass of the wire $\ll$ mass of the steel ball, the radius of the steel ball $\ll$ length of the wire [ 0.5 pts ]
flexibility of the wire, air friction, etc [ 0.1 pts, only when one of the two major points above is not given]
3. [Total 1.0 pts] Data sample from simple pendulum experiment $\#$ of cycle $\geq 20$ [0.2 pts.] , difference in $\mathrm{T} \geq 0.01 \mathrm{~s}$ [0.4 pts], \# of data $\geq 4$ [0.4 pts]

| No. | $\mathrm{t}(\mathrm{s})$ for 50 cycles | Period, T (s) | Scale marked on the <br> wire (arbitrary scale) |
| :---: | :---: | :---: | :---: |
| 1 | 91.47 | 1.83 | 200 |
| 2 | 89.09 | 1.78 | 150 |
| 3 | 86.45 | 1.73 | 100 |
| 4 | 83.8 | 1.68 | 50 |

4. [Total 0.5 pts$]$

| No. | Period, T (s) | Scale marked on the wire <br> (arbitrary scale) | $\mathrm{T}^{2}\left(\mathrm{~s}^{2}\right)$ |
| :---: | :---: | :---: | :---: |
| 1 | 1.83 | 200 | 3.35 |
| 2 | 1.78 | 150 | 3.17 |
| 3 | 1.73 | 100 | 2.99 |
| 4 | 1.68 | 50 | 2.81 |

The plot of $\mathrm{T}^{2}$ vs scale marked on the wire:


Scale marked on the wire (arbitrary scale)
5. Determination of the smallest unit of the arbitrary scale in term of mm [Total 1.5 $\mathrm{pts}]$

$$
\begin{aligned}
& T_{o s c_{1}}^{2}=\frac{4 \pi^{2}}{g} L_{1}, \quad T_{o s c_{2}}^{2}=\frac{4 \pi^{2}}{g} L_{2} \\
& \left(T_{o s c_{1}}^{2}-T_{o s c_{2}}^{2}\right)=\frac{4 \pi^{2}}{g} L_{1}-L_{2}=\frac{4 \pi^{2}}{g} \Delta L
\end{aligned}
$$

$\Delta L=\frac{g}{4 \pi^{2}}\left(T_{\text {osc }_{1}}^{2}-T_{\text {osc }_{2}}^{2}\right)$ or other equivalent expression

| No. |  | Calculated $\Delta \mathrm{L}(\mathrm{m})$ | $\Delta \mathrm{L}$ in arbitrary <br> scale | Values of smallest <br> unit of arbitrary <br> scale $(\mathrm{mm})$ |
| :---: | :---: | :---: | :---: | :---: |
| 1. | $\mathrm{~T}_{1}{ }^{2}-\mathrm{T}_{2}{ }^{2}=0.171893 \mathrm{~s}^{2}$ | 0.042626 | 50 | 0.85 |
| 2. | $\mathrm{~T}_{1}{ }^{2}-\mathrm{T}_{3}{ }^{2}=0.357263 \mathrm{~s}^{2}$ | 0.088595 | 100 | 0.89 |
| 3. | $\mathrm{~T}_{1}{ }^{2}-\mathrm{T}_{4}{ }^{2}=0.537728 \mathrm{~s}^{2}$ | 0.133347 | 150 | 0.89 |
| 4. | $\mathrm{~T}_{2}{ }^{2}-\mathrm{T}_{3}{ }^{2}=0.18537 \mathrm{~s}^{2}$ | 0.045968 | 50 | 0.92 |
| 5. | $\mathrm{~T}_{2}{ }^{2} \mathrm{~T}_{4}{ }^{2}=0.365835 \mathrm{~s}^{2}$ | 0.09072 | 100 | 0.91 |
| 6. | $\mathrm{~T}_{3}{ }^{2}-\mathrm{T}_{4}{ }^{2}=0.180465 \mathrm{~s}^{2}$ | 0.044752 | 50 | 0.90 |

The average value of smallest unit of arbitrary scale, $\bar{l}=0.89 \mathrm{~mm}$

The estimated error induced by the measurement: [0.5 pts]

| No. | Values of smallest <br> unit of arbitrary <br> scale $(\mathrm{mm})$ | $(l-\bar{l})$ | $(l-\bar{l})^{2}$ |
| :---: | :---: | :---: | :---: |
| 1. | 0.85 | -0.04 | 0.0016 |
| 2. | 0.89 | 0 | 0 |
| 3. | 0.89 | 0 | 0 |
| 4. | 0.92 | 0.03 | 0.0009 |
| 5. | 0.91 | 0.02 | 0.0004 |
| 6. | 0.90 | 0.01 | 0.0001 |

And the standard deviation is:

$$
\Delta l=\sqrt{\frac{\sum_{i=1}^{6}(l-\bar{l})^{2}}{N-1}}=\sqrt{\frac{0.003}{5}}=0.02 \mathrm{~mm}
$$

other legitimate methods may be used

## PART B

1. The experimental set up:[Total $\mathbf{1 . 0} \mathbf{~ p t s}]$
[0.2 pts]
[0.2 pts]

2. Derivation of equation relating the quantities time $t$, current $I$, and water level difference $\Delta h$ : :[Total 1.5 pts]
$I=\frac{\Delta Q}{\Delta t}$
From the reaction: $2 \mathrm{H}^{+}+2 \mathrm{e} \longrightarrow \mathrm{H}_{2}$, the number of molecules produced in the process $(\Delta N)$ requires the transfer of electric change is $\Delta \mathrm{Q}=2 \mathrm{e} \Delta \mathrm{N}: \quad$ [0.2 pts]

$$
\begin{align*}
I & =\frac{\Delta \mathrm{N} 2 \mathrm{e}}{\Delta \mathrm{t}}  \tag{0.5pts}\\
P \Delta \mathrm{~V} & =\Delta \mathrm{N} \mathrm{k}_{\mathrm{B}} \mathrm{~T}  \tag{0.5pts}\\
& =\frac{I \Delta t}{2 \mathrm{e}} \mathrm{k}_{\mathrm{B}} \mathrm{~T} \\
\mathrm{P} \Delta \mathrm{~h}\left(\pi r^{2}\right) & =\frac{I \Delta t}{2} \frac{\mathrm{k}_{\mathrm{B}}}{e} \mathrm{~T}  \tag{0.2pts}\\
I \Delta t & =\frac{\mathrm{e}}{\mathrm{k}_{\mathrm{B}}} \frac{2 P\left(\pi \mathrm{r}^{2}\right)}{T} \Delta \mathrm{~h} \tag{0.1pts}
\end{align*}
$$

3. The experimental data: [ Total $1.0 \mathbf{p t s}$ ]

| No. | $\Delta \mathrm{h}$ (arbitrary <br> scale) | $\mathrm{I}(\mathrm{mA})$ | $\Delta \mathrm{t}(\mathrm{s})$ |
| :---: | :---: | :---: | :---: |
| 1 | 12 | 4.00 | 1560.41 |
| 2 | 16 | 4.00 | 2280.61 |
| 3 | 20 | 4.00 | 2940.00 |
| 4 | 24 | 4.00 | 3600.13 |

- The circumference $\phi$, of the test tube $=46$ arbitrary scale
[0.3 pts]
- The chosen values for $\Delta h$ ( $\geq 4$ scale unit) for acceptable error due to uncertainty of the water level reading and for $I(\leq 4 \mathrm{~mA})$ for acceptable disturbance [ 0.3 pts ]
- \# of data $\geq 4$

The surrounding condition $(T, P)$ in which the experimental data given above taken:

$$
\begin{aligned}
& T=300 \mathrm{~K} \\
& P=1.0010^{5} \mathrm{~Pa}
\end{aligned}
$$

4. Determination the value of $\mathrm{e} / \mathrm{k}_{\mathrm{B}}$ [Total $\mathbf{1 . 5} \mathbf{~ p t s ]}$

| No. | $\Delta \mathrm{h}$ (arbitrary <br> scale) | $\Delta \mathrm{h}(\mathrm{mm})$ | $\mathrm{I}(\mathrm{mA})$ | $\Delta \mathrm{t}(\mathrm{s})$ | $\mathrm{I} \Delta \mathrm{t}(\mathrm{C})$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 12 | 10.68 | 4.00 | 1560.41 | 6241.64 |
| 2 | 16 | 14.24 | 4.00 | 2280.61 | 9120.48 |
| 3 | 20 | 17.80 | 4.00 | 2940.00 | 11760.00 |
| 4 | 24 | 21.36 | 4.00 | 3600.13 | 14400.52 |

Plot of $\mathrm{I} \Delta \mathrm{t}$ vs $\Delta \mathrm{h}$ from the data listed above


The slope obtained from the plot is 763.94;

$$
\begin{aligned}
& \frac{\mathrm{e}}{\mathrm{k}_{\mathrm{B}}}=\frac{763.94 \times 300 \times \pi}{2 \times 10^{5} \times\left(23 \times 0.89 \times 10^{-3} \times 0.82\right)^{2}}=1.28 \times 10^{4} \text { Coulomb K/J } \\
& {[1.0 \mathrm{pts}]}
\end{aligned}
$$

Alternatively [the same credit points]

| No. | $\Delta \mathrm{h}(\mathrm{mm})$ | $\mathrm{I} \Delta \mathrm{t}(\mathrm{C})$ | Slope | $\mathrm{e} / \mathrm{k}_{\mathrm{b}}$ |
| :---: | :---: | :---: | :---: | ---: |
| 1 | 10.68 | 6241.64 | 584.4232 | 9774.74 |
| 2 | 14.24 | 9120.48 | 640.4831 | 10712.37 |
| 3 | 17.80 | 11760.00 | 660.6742 | 11050.07 |
| 4 | 21.36 | 14400.52 | 674.1816 | 11275.99 |

Average of $\mathrm{e} / \mathrm{k}_{\mathrm{b}}=1.07 \times 10^{4} \quad$ Coulomb K $/ \mathrm{J}$ [1.0 pts]

| No. | $\mathrm{e} / \mathrm{k}_{\mathrm{b}}$ | difference | Square <br> difference |
| ---: | ---: | ---: | ---: |
| 1 | 9774.74 | -928.55 | 862205.5 |
| 2 | 10712.37 | 9.077117 | 82.39405 |
| 3 | 11050.07 | 346.7808 | 120256.9 |
| 4 | 11275.99 | 572.6996 | 327984.9 |

Estimated error
The standard deviation obtained is $0.66 \times 10^{3} \quad$ Coulomb K/J,
Other legitimate measures of estimated error may be also used

## SOLUTION OF EXPERIMENT PROBLEM 2

1. The optical components are [total 1.5 pts$]$ :

| no. 1 | Diffraction grating | $[\mathbf{0 . 5} \boldsymbol{p t s}]$ |
| :--- | :--- | :--- |
| no. 2 | Diffraction grating | $[\mathbf{0 . 5} \boldsymbol{p t s}]$ |
| no. 3 | Plan-parallel plate | $[\mathbf{0 . 5 ~ p t s}]$ |

2. Cross section of the box [total 1.5 pts$]$ :

3. Additional information [total $\mathbf{1 . 0} \mathbf{~ p t s}]$ :


Distance of the grating (no.1) to the left wall is practically zero [0.2 pts]

Lines of grating no. 1 is at right angle to the slit
[0.3 pts]

Distance of the grating (no.2) to the right wall is practically zero [0.2 pts]

Lines of grating no. 2
is parallel to the slit
[0.3 pts]
4. Diffraction grating [total 2.0 pts :


Path length difference:

$$
\Delta=d \sin \theta, \quad d=\text { spacing of the grating }
$$

Diffraction order:

$$
\Delta=m \lambda, \quad m=\text { order number }
$$

Hence, for the first order ( $m=1$ ):

$$
\sin \theta=\lambda / d
$$

[0.4 pts]

Observation data:

| $\tan \theta$ | $\theta$ | $\sin \theta$ |
| :--- | :--- | :--- |

$\begin{array}{lll}0.34 & 18.78^{0} & 0.3219\end{array}$
$\begin{array}{cccc}0.32 & 17.74^{0} & 0.3048 & \text { number of data } \geq 3 \\ 0.32 & 17.74^{0} & 0.3048 & {[\mathbf{0 . 5 ~ p t s}]}\end{array}$

| Name of component no.1 | Specification |
| :---: | :---: |
| Diffraction grating | Spacing $=2.16 ~ \mu \mathrm{~m}$ <br> Lines at right angle to the slit |

[0.4 pts]
[0.1 pts]

Note: true value of grating spacing is $2.0 \mu \mathrm{~m}$, deviation of the result $\leq 10 \%$
5. Diffraction grating [total 2.0 pts :

For the derivation of the formula, see nr. 4 above.
[1.0 pts]

Observation data:

| $\tan \theta$ | $\theta$ | $\sin \theta$ |  |
| :--- | :--- | :--- | :---: |
| 1.04 | $46.12^{0}$ | 0.7208 |  |
| 0.96 | $43.83^{0}$ | 0.6925 | number of data $\geq 3$ |
| 1.08 | $47.20^{\circ}$ | 0.7330 | [0.5 pts] |


| Name of component no.2 | Specification |
| :---: | :---: |
| Diffraction grating | Spacing $=0.936 ~$ <br>  <br> Lines parallel to the slit |

[0.4 pts]
[0.1 pts]

Note: true value of grating spacing is $1.0 \mu \mathrm{~m}$, deviation of the result $\leq 10 \%$


Snell's law:

$$
\sin \varphi=n \sin \varphi^{\prime}, \quad \varphi^{\prime}=\angle \mathrm{BAC}
$$

Path length inside the plate:

$$
\mathrm{AC}=\mathrm{AB} / \cos \varphi^{\prime}, \quad \mathrm{AB}=h=\text { plate thickness }
$$

Beam displacement:

$$
\mathrm{CD}=t=\mathrm{AC} \sin \angle \mathrm{CAD}, \quad \angle \mathrm{CAD}=\varphi-\varphi^{\prime}
$$

Hence:

$$
t=h \sin \varphi\left[1-\cos \varphi /\left(n^{2}-\sin ^{2} \varphi\right)^{1 / 2}\right] \quad[0.6 \boldsymbol{p t s}]
$$

## Observation data:

| $\varphi$ | $t$ |  |
| :--- | :--- | :--- |
| 0 | 0 | (angle between beam and axis $\left.49^{\circ}\right)$ |
| $49^{\circ}$ | 7.3 arbitrary scale |  |


| Name of component no.3 | Specification |
| :---: | :--- |
| Plane-parallel plate | Thickness $=17.9 \mathrm{~mm}$ <br> Angle to the axis of the box $49^{\circ}$ |

Note: - true value of plate thickness is 20 mm

- true value of angle to the axis of the box is $52^{\circ}$
- deviation of the results $\leq 20 \%$.


## Theoretical Question 1

## A Swing with a Falling Weight

A rigid cylindrical rod of radius $R$ is held horizontal above the ground. With a string of negligible mass and length $L(L>2 \pi R)$, a pendulum bob of mass $m$ is suspended from point $A$ at the top of the rod as shown in Figure 1a. The bob is raised until it is level with $A$ and then released from rest when the string is taut. Neglect any stretching of the string. Assume the pendulum bob may be treated as a mass point and swings only in a plane perpendicular to the axis of the rod. Accordingly, the pendulum bob is also referred to as the particle. The acceleration of gravity is $\vec{g}$.


Figure 1a

Let $O$ be the origin of the coordinate system. When the particle is at point $P$, the string is tangential to the cylindrical surface at $Q$. The length of the line segment $Q P$ is called $s$. The unit tangent vector and the unit radial vector at $Q$ are given by $\hat{t}$ and $\hat{r}$, respectively. The angular displacement $\theta$ of the radius $O Q$, as measured counterclockwise from the vertical $x$-axis along $O A$, is taken to be positive.

When $\theta=0$, the length $s$ is equal to $L$ and the gravitational potential energy $U$ of the particle is zero. As the particle moves, the instantaneous time rates of change of $\theta$ and $s$ are given by $\dot{\theta}$ and $\dot{s}$, respectively.

Unless otherwise stated, all the speeds and velocities are relative to the fixed point $O$.

## Part A

In Part A, the string is taut as the particle moves. In terms of the quantities introduced above (i.e., $s, \theta, \dot{s}, \dot{\theta}, R, L, g, \hat{t}$ and $\hat{r}$ ), find:
(a)The relation between $\dot{\theta}$ and $\dot{s}$.
(b)The velocity $\vec{v}_{Q}$ of the moving point $Q$ relativeto $O$.
(c)The particle's velocity $\bar{v}^{\prime}$ relative to the moving point $Q$ when it is at $P$
(d)The particle's velocity $\vec{v}$ relative to $O$ when it is at $P$.
(e) The $\hat{t}$-component of the particle's acceleration relative to $O$ when it is at $P$.
(0.7 point)
(f) The particle's gravitational potential energy $U$ when it is at $P$.
(0.5 point)
(g)The speed $v_{m}$ of the particle at the lowest point of its trajectory.

## Part B

In Part B, the ratio $L$ to $R$ has the following value:

$$
\frac{L}{R}=\frac{9 \pi}{8}+\frac{2}{3} \cot \frac{\pi}{16}=3.534+3.352=6.886
$$

(h)What is the speed $v_{s}$ of the particle when the string segment from $Q$ to $P$ is both straight and shortest in length? (in terms of $g$ and $R$ )
(i) What is the speed $v_{H}$ of the particle at its highest point $H$ when it has swung to the other side of the rod? (in terms of $g$ and $R$ )

## Part C

In Part C, instead of being suspended from $A$, the pendulum bob of mass $m$ is connected by a string over the top of the rod to a heavier weight of mass $M$, as shown in Figure 1b. The weight can also be treated as a particle.


Initially, the bob is held stationary at the same level as $A$ so that, with the weight hanging below $O$, the string is taut with a horizontal section of length $L$. The bob is then released from rest and the weight starts falling. Assume that the bob remains in a vertical plane and can swing past the falling weight without any interruption.

The kinetic friction between the string and the rod surface is negligible. But the static friction is assumed to be large enough so that the weight will remain stationary once it has come to a stop (i.e. zero velocity).
(j) Assume that the weight indeed comes to a stop after falling a distance $D$ and that $(L-D) \gg R$. If the particle can then swing around the rod to $\theta=2 \pi$ while both segments of the string free from the rod remain straight, the ratio $\alpha=D / L$ must not be smaller than a critical value $\alpha_{c}$. Neglecting terms of the order $R / L$ or higher, obtain an estimate on $\alpha_{c}$ in terms of $M / m$.
(3.4 points)

## Answer Sheet <br> Theoretical Question 1 <br> A Swing with a Falling Weight

(a) The relation between $\dot{\theta}$ and $\dot{s}$ is
$\square$
(b) The velocity of the moving point $Q$ relative to $O$ is

```
\mp@subsup{\stackrel{\rightharpoonup}{Q}}{Q}{}=
```

(c) When at $P$, the particle's velocity relative to the moving point $Q$ is

```
\mp@subsup{\vec{v}}{}{\prime}}
```

(d) When at $P$, the particle's velocity relative to $O$ is

```
v =
```

(e) When at $P$, the $\hat{t}$-component of the particle's acceleration relative to $O$ is
$\square$
(f) When at $P$, the particle's gravitational potential energy is

$$
U=
$$

(g) The particle's speed when at the lowest point of its trajectory is

$$
v_{m}=
$$

(h) When line segment $Q P$ is straight with the shortest length, the particle's speed is (Give expression and value in terms of $g$ and $R$ )

$$
v_{s}=
$$

(i) At the highest point, the particle's speed is (Give expression and value in terms of $g$ and $R$ )

$$
v_{H}=
$$

(j) In terms of the mass ratio $M / m$, the critical value $\alpha_{\mathrm{c}}$ of the ratio $D / L$ is

$$
\alpha_{\mathrm{c}}=
$$

## Theoretical Question 2

## A Piezoelectric Crystal Resonator under an Alternating Voltage

Consider a uniform rod of unstressed length $\ell$ and cross-sectional area $A$ (Figure 2a). Its length changes by $\Delta \ell$ when equal and opposite forces of magnitude $F$ are applied to its ends faces normally. The stress $T$ on the end faces is defined to be F/A. The fractional change in its length, i.e., $\Delta \ell / \ell$, is called the strain $S$ of the rod. In terms of stress and strain, Hooke's law may be expressed as

$$
\begin{equation*}
T=Y S \quad \text { or } \quad \frac{F}{A}=Y \frac{\Delta \ell}{\ell} \tag{1}
\end{equation*}
$$

where $Y$ is called the Young's modulus of the rod material. Note that a compressive stress $T$ corresponds to $F<0$ and a decrease in length (i.e., $\Delta \ell<0$ ). Such a stress is thus negative in value and is related to the pressure $p$ by $T=-p$.

For a uniform rod of density $\rho$, the speed of propagation of longitudinal waves (i.e., sound speed) along the rod is given by

$$
\begin{equation*}
u=\sqrt{Y / \rho} \tag{2}
\end{equation*}
$$



Figure 2a
The effect of damping and dissipation can be ignored in answering the following questions.

## Part A: mechanical properties

A uniform rod of semi-infinite length, extending from $x=0$ to $\infty$ (see Figure 2b), has a density $\rho$. It is initially stationary and unstressed. A piston then steadily exerts a small pressure $p$ on its left face at $x=0$ for a very short time $\Delta t$, causing a pressure wave to propagate with speed $u$ to the right.


Figure 2b


Figure 2c
(a)If the piston causes the rod's left face to move at a constant velocity $v$ (Figure 2b), what are
the strain $S$ and pressure $p$ at the left face during the time $\Delta t$ ? Answers must be given in terms of $\rho, u$, and $v$ only.
(b)Consider a longitudinal wave traveling along the $x$ direction in the rod. For a cross section at $x$ when the rod is unstressed (Figure 2c), let $\xi(x, t)$ be its displacement at time $t$ and assume

$$
\begin{equation*}
\xi(x, t)=\xi_{0} \sin k(x-u t) \tag{3}
\end{equation*}
$$

where $\xi_{0}$ and $k$ are constants. Determine the corresponding velocity $v(x, t)$, strain $S(x, t)$, and pressure $p(x, t)$ as a function of $x$ and $t$.

## Part B: electromechanical properties (including piezoelectric effect)

Consider a quartz crystal slab of length $b$, thickness $h$, and width $w$ (Figure 2d). Its length and thickness are along the $x$-axis and $z$-axis. Electrodes are formed by thin metallic coatings at its top and bottom surfaces. Electrical leads that also serve as mounting support (Figure 2e) are soldered to the electrode's centers, which may be assumed to be stationary for longitudinal oscillations along the $x$ direction.


Figure 2d


Figure 2e

The quartz crystal under consideration has a density $\rho$ of $2.65 \times 10^{3} \mathrm{~kg} / \mathrm{m}^{3}$ and Young's modulus $Y$ of $7.87 \times 10^{10} \mathrm{~N} / \mathrm{m}^{2}$. The length $b$ of the slab is 1.00 cm and the width $w$ and height $h$ of the slab are such that $h \ll w$ and $w \ll b$. With switch $K$ left open, we assume only longitudinal modes of standing wave oscillation in the $x$ direction are excited in the quartz slab.

For a standing wave of frequency $f=\omega / 2 \pi$, the displacement $\xi(x, t)$ at time $t$ of a cross section of the slab with equilibrium position $x$ may be written as

$$
\begin{equation*}
\xi(x, t)=2 \xi_{0} g(x) \cos \omega t, \quad(0 \leq x \leq b) \tag{4a}
\end{equation*}
$$

where $\xi_{0}$ is a positive constant and the spatial function $g(x)$ is of the form

$$
\begin{equation*}
g(x)=B_{1} \sin k\left(x-\frac{b}{2}\right)+B_{2} \cos k\left(x-\frac{b}{2}\right) \tag{4b}
\end{equation*}
$$

$\mathrm{g}(x)$ has the maximum value of one and $k=\omega / u$. Keep in mind that the centers of the electrodes are stationary and the left and right faces of the slab are free and must have zero stress (or pressure).
(c)Determine the values of $B_{1}$ and $B_{2}$ in Eq. (4b) for a longitudinal standing wave in the quartz slab.
(d)What are the two lowest frequencies at which longitudinal standing waves may be excited in the quartz slab?
(1.2 point)

The piezoelectric effect is a special property of a quartz crystal. Compression or dilatation of the crystal generates an electric voltage across the crystal, and conversely, an external voltage applied across the crystal causes the crystal to expand or contract depending on the polarity of the voltage. Therefore, mechanical and electrical oscillations can be coupled and made to resonate through a quartz crystal.

To account for the piezoelectric effect, let the surface charge densities on the upper and lower electrodes be $-\sigma$ and $+\sigma$, respectively, when the quartz slab is under an electric field $E$ in the $z$ direction. Denote the slab's strain and stress in the $x$ direction by $S$ and $T$, respectively. Then the piezoelectric effect of the quartz crystal can be described by the following set of equations:

$$
\begin{array}{r}
S=(1 / Y) T+d_{p} E \\
\sigma=d_{p} T+\varepsilon_{T} E \tag{5b}
\end{array}
$$

where $1 / Y=1.27 \times 10^{-11} \mathrm{~m}^{2} / \mathrm{N}$ is the elastic compliance (i.e., inverse of Young's modulus) at constant electric field and $\varepsilon_{T}=4.06 \times 10^{-11} \mathrm{~F} / \mathrm{m}$ is the permittivity at constant stress, while $d_{p}$ $=2.25 \times 10^{-12} \mathrm{~m} / \mathrm{V}$ is the piezoelectric coefficient.

Let switch $K$ in Fig. 2d be closed. The alternating voltage $V(t)=V_{m} \cos \omega t$ now acts across the electrodes and a uniform electric field $E(t)=V(t) / h$ in the $z$ direction appears in the quartz slab. When a steady state is reached, a longitudinal standing wave oscillation of angular frequency $\omega$ appears in the slab in the $x$ direction.

With $E$ being uniform, the wavelength $\lambda$ and the frequency $f$ of a longitudinal standing wave in the slab are still related by $\lambda=u / f$ with $u$ given by Eq. (2). But, as Eq. (5a) shows, $T$ = YS is no longer valid, although the definitions of strain and stress remain unchanged and the end faces of the slab remain free with zero stress.
(e)Taking Eqs. (5a) and (5b) into account, the surface charge density $\sigma$ on the lower electrode as a function of $x$ and $t$ is of the form,

$$
\sigma(x, t)=\left[D_{1} \cos k\left(x-\frac{b}{2}\right)+D_{2}\right] \frac{V(t)}{h}
$$

where $k=\omega / u$. Find the expressions for $D_{1}$ and $D_{2}$.
(f) The total surface charge $Q(t)$ on the lower electrode is related to $V(t)$ by

$$
\begin{equation*}
Q(t)=\left[1+\alpha^{2}\left(\frac{2}{k b} \tan \frac{k b}{2}-1\right)\right] C_{0} V(t) \tag{6}
\end{equation*}
$$

Find the expression for $C_{0}$ and the expression and numerical value of $\alpha^{2}$.

## [Answer Sheet] Theoretical Question 2

## A Piezoelectric Crystal Resonator under an Alternating Voltage

## Wherever requested, give each answer as analytical expressions followed by

 numerical values and units. For example: area of a circle $A=\pi r^{2}=1.23 \mathrm{~m}^{2}$.(a) The strain $S$ and pressure $p$ at the left face are (in terms of $\rho$, $u$, and $v$ )

| $S=$ |
| :--- |
| $p=$ |

(b) The velocity $v(x, t)$, strain $S(x, t)$, and pressure $p(x, t)$ are

| $v(x, t)=$ |
| :--- |
| $S(x, t)=$ |
| $p(x, t)=$ |

(c) The values of $B_{1}$ and $B_{2}$ are

| $B_{1}=$ |
| :--- |
| $B_{2}=$ |

(d) The lowest two frequencies of standing waves are (expression and value)

| The Lowest |
| :--- |
| The Second Lowest |

(e) The expressions of $D_{1}$ and $D_{2}$ are

| $D_{1}=$ |
| :--- |
| $D_{2}=$ |

(f) The constants $\alpha^{2}$ (expression and value) and $C_{0}$ are (expression only)

| $\alpha^{2}=$ |
| :--- |
| $C_{0}=$ |

## Theoretical Question 3

## Part A

## Neutrino Mass and Neutron Decay

A free neutron of mass $m_{n}$ decays at rest in the laboratory frame of reference into three non-interacting particles: a proton, an electron, and an anti-neutrino. The rest mass of the proton is $m_{p}$, while the rest mass of the anti-neutrino $m_{\nu}$ is assumed to be nonzero and much smaller than the rest mass of the electron $m_{e}$. Denote the speed of light in vacuum by $c$. The measured values of mass are as follows:

$$
m_{n}=939.56563 \mathrm{MeV} / c^{2}, m_{p}=938.27231 \mathrm{MeV} / c^{2}, m_{e}=0.5109907 \mathrm{MeV} / c^{2}
$$

In the following, all energies and velocities are referred to the laboratory frame. Let $E$ be the total energy of the electron coming out of the decay.
(a) Find the maximum possible value $E_{\text {max }}$ of $E$ and the speed $v_{\mathrm{m}}$ of the anti-neutrino when $E$ $=E_{\text {max }}$. Both answers must be expressed in terms of the rest masses of the particles and the speed of light. Given that $m_{v}<7.3 \mathrm{eV} / c^{2}$, compute $E_{\max }$ and the ratio $v_{\mathrm{m}} / c$ to 3 significant digits.

## Part B

## Light Levitation

A transparent glass hemisphere with radius $R$ and mass $m$ has an index of refraction $n$. In the medium outside the hemisphere, the index of refraction is equal to one. A parallel beam of monochromatic laser light is incident uniformly and normally onto the central portion of its planar surface, as shown in Figure 3. The acceleration of gravity $\bar{g}$ is vertically downwards. The radius $\delta$ of the circular cross-section of the laser beam is much smaller than $R$. Both the glass hemisphere and the laser beam are axially symmetric with respect to the $z$-axis.

The glass hemisphere does not absorb any laser light. Its surface has been coated with a thin layer of transparent material so that reflections are negligible when light enters and leaves the glass hemisphere. The optical path traversed by laser light passing through the non-reflecting surface layer is also negligible.
(b) Neglecting terms of the order $(\delta / R)^{3}$ or higher, find the laser power $P$ needed to balance the weight of the glass hemisphere.
Hint: $\cos \theta \approx 1-\theta^{2} / 2$ when $\theta$ is much smaller than one.


## [Answer Sheet] Theoretical Question 3

Wherever requested, give each answer as analytical expressions followed by numerical values and units. For example: area of a circle $A=\pi r^{2}=1.23 \mathrm{~m}^{2}$.

## Neutrino Mass and Neutron Decay

(a) (Give expressions in terms of rest masses of the particles and the speed of light)

The maximum energy of the electron is (expression and value)

$$
E_{\max }=
$$

The ratio of anti-neutrino's speed at $E=E_{\text {max }}$ to $c$ is (expression and value)
$\square$

## Light Levitation

(b) The laser power needed to balance the weight of the glass hemisphere is
$P=\square$

# The 34th International Physics Olympiad 

Taipei, Taiwan

## Experimental Competition

## Wednesday, August 6, 2003

## Time Available : 5 hours

## Please Read This First:

1. Use only the pen provided.
2. Use only the front side of the answer sheets and paper.
3. In your answers please use as little text as possible; express yourself primarily in equations, numbers and figures. If the required result is a numerical number, underline your final result with a wavy line.
4. Write on the blank sheets of paper the results of your measurements and whatever else you consider is required for the solution of the question and that you wish to be marked.
5. It is absolutely essential that you enter in the boxes at the top of each sheet of paper used your Country and your student number [Student No.]. In addition, on the blank sheets of paper used for each question, you should enter the question number [Question No. : e.g. A-(1)], the progressive number of each sheet [Page No.] and the total number of blank sheets that you have used and wish to be marked for each question [Total No. of pages]. If you use some blank sheets of paper for notes that you do not wish to be marked, put a large cross through the whole sheet and do not include them in your numbering.
6. At the end of the exam please put your answer sheets and graphs in order.
7. Error bars on graphs are only needed in part A of the experiment.
8. Caution: Do not look directly into the laser beam. You can damage your eyes!!

## Apparatuses and materials

1. Available apparatuses and materials are listed in the following table:

| Item | Apparatus \& material | Quantity | Item | Apparatus \& material | Quantity |
| :---: | :---: | :---: | :---: | :---: | :---: |
| A | Photodetector (PD) | 1 | I | Batteries | 2 |
| B | Polarizers with <br> Rotary mount | 2 | J | Battery box | 1 |
| C | $90^{\circ} \mathrm{TN}$-LC cell (yellow wires) with rotary LC mount | 1 | K | Optical bench | 1 |
| D | Function generator | 1 | L | Partially transparent papers | 2 |
| E | Laser diode (LD) | 1 | M | Ruler | 1 |
| F | Multimeters | 2 | N | White tape * <br> (for marking on apparatus) | 1 |
| G | Parallel LC cell <br> (orange wires) | 1 | O | Scissors | 1 |
| H | Variable resistor | 1 | P | Graph papers | 10 |

* Do not mark directly on apparatus. When needed, stick a piece of the white tape on the parts and mark on the white tape.


Fig. 1
2. Instructions for the multimeter:

- "DC/AC" switch for selecting DC or AC measurement.
- Use the "V $\Omega$ " and the "COM" inlets for voltage and resistance measurements.
- Use the "mA" and the "COM" inlets for small current measurements. The display then shows the current in milliamperes.
- Use the function dial to select the proper function and measuring range. "V" is for voltage measurement, "A" is for current measurement and " $\Omega$ "is for resistance measurement.


Fig. 2

## 3. Instructions for the Function Generator:

- The power button may be pressed for "ON" and pressed again for "OFF"
- Select the frequencies range, and press the proper button.
- The frequency is shown on the digital display.
- Use the coarse and the fine frequency adjusting knobs to tune the proper frequency.
- Select the square-wave form by pressing the left most waveform button.
- Use the amplitude-adjusting knob to vary the output voltage.


Fig. 3

## Part A: Optical Properties of Laser Diode

## I. Introduction

1. Laser Diode

The light source in this experiment is a laser diode which emits laser light with wavelength 650 nm . When the current of the laser diode (LD) is greater than the threshold current, the laser diode can emit monochromatic, partially polarized and coherent light. When the current in the laser diode is less than the threshold, the emitted light intensity is very small. At above the threshold current, the light intensity increases dramatically with the current and keeps a linear relationship with the current. If the current increases further, then the increasing rate of the intensity with respect to the current becomes smaller because of the higher temperature of the laser diode. Therefore, the optimal operating current range for the laser diode is the region where the intensity is linear with the current. In general, the threshold current $I_{t h}$ is defined as the intersection point of the current axis with the extrapolation line of the linear region.
Caution: Do not look directly into the laser beam. You can damage your eyes!!

## 2. Photodetector

The photodetector used in this experiment consists of a photodiode and a current amplifier. When an external bias voltage is applied on the photodiode, the photocurrent is generated by the light incident upon the diode. Under the condition of a constant temperature and monochromatic incident light, the photocurrent is proportional to the light intensity. On the other hand, the current amplifier is utilized to transfer the photocurrent into an output voltage. There are two transfer ratios in our photodetector - high and low gains. In our experiment, only the low gain is used. However, because of the limitation of the photodiode itself, the output voltage would go into saturation at about 8 Volts if the light intensity is too high and the photodiode cannot operate properly any more. Hence the appropriate operating range of the photodetector is when the output voltage is indeed proportional to the light intensity. If the light intensity is too high so that the photodiode reaches the saturation, the reading of the photodetector can not correctly represent the incident light intensity.

## II. Experiments and procedures

## Characteristics of the laser diode \& the photodetector

In order to make sure the experiments are done successfully, the optical alignment of
light rays between different parts of an experimental setup is crucial. Also the light source and the detector should be operated at proper condition. Part A is related to these questions and the question of the degree of polarization.

1. Mount the laser diode and photodetector in a horizontal line on the optical bench, as shown in Fig. 4. Connect the variable resistor, battery set, ampere meter, voltage meter, laser diode and photodetector according to Fig. 5. Adjust the variable resistor so that the current passing through LD is around 25 mA and the laser diode emits laser light properly. Choose the low gain for the photodetector. Align the laser diode and the photodetector to make the laser light level at the small hole on the detector box and the reading of the photodetector reaches a maximum value.
Caution: Do not let the black and the red leads of the battery contact with each other to avoid short circuit.


Fig. 4 Optical setup (LD : laser diode; PD : photodetector).


Fig. 5 Equivalent circuit for the connection of the laser diode.
2. Use the output voltage of the photodetector to represent the laser light intensity $\mathcal{J}$. Adjust the variable resistor to make the current $I$ of the laser diode varying from zero to a maximum value and measure the $\mathcal{J}$ as $I$ increases. Be sure to choose appropriate current increment in the measurement.

Question A-(1) (1.5 point)
Measure, tabulate, and plot the $\mathcal{J}$ vs. I curve.

Question A-(2) (3.5 points)
Estimate the maximum current $I_{m}$ with uncertainty in the linear region of the $\mathcal{J}$ vs. $I$ curve. Mark the linear region on the $\mathcal{J}-I$ curve figure by using arrows $(\downarrow)$ and determine the threshold current $I_{t h}$ with uncertainty.
3. Choose the current of the laser diode as $I_{t h}+2\left(I_{m}-I_{t h}\right) / 3$ to make sure the laser diode and photodetector are operated well.
4. To prepare for the part B experiment: Mount a polarizer on the optical bench close to the laser diode as shown in Fig. 6. Make sure the laser beam passing through the center portion of the polarizer. Adjust the polarizer so that the incident laser beam is perpendicular to the plane of the polarizer. (Hint: You can insert a piece of partially transparent paper as a test screen to check if the incident and reflected light spots coincide with each other.)


Fig. 6 Alignment of the polarizer ( P : polarizer).
5. Keep the current of the laser diode unchanged, mount a second piece of polarizer on the optical bench and make sure proper alignment is accomplished, i.e., set up the source, detector and polarizers in a straight line and make sure each polarizer plane is perpendicular to the light beam.

# Part B Optical Properties of Nematic Liquid Crystal : <br> Electro-optical switching characteristic of $90^{\circ} \mathrm{TN}$ LC cell 

## I. Introduction

## 1. Liquid Crystal

Liquid crystal (LC) is a state of matter that is intermediate between the crystalline solid and the amorphous liquid. The nematic LCs are organic compounds consist of long-shaped needle-like molecules. The orientation of the molecules can be easily aligned and controlled by applying an electrical field. Uniform or well prescribed orientation of the LC molecules is required in most LC devices. The structure of the LC cell used in this experiment is shown in Fig 7. Rubbing the polyimide film can produce a well-aligned preferred orientation for LC molecules on substrate surfaces, thus due to the molecular interaction the whole slab of LC can achieve uniform molecular orientation. The local molecular orientation is called the director of LC at that point.

The LC cell exhibits the so-called double refraction phenomenon with two principal refractive indices. When light propagates along the direction of the director, all polarization components travel with the same speed $v_{o}=c / n_{0}$, where $n_{o}$ is called the ordinary index of refraction. This propagation direction (direction of the director) is called the optic axis of the LC cell. When a light beam propagates in the direction perpendicular to the optic axis, in general, there are two speeds of propagation. The electric field of the light polarized perpendicular (or parallel) to the optic axis travels with the speed of $v_{o}=c / n_{0}$ (or $v_{e}=c / n_{e}$, where $n_{e}$ is called the extraordinary index of refraction). The birefringence (optical anisotropy) is defined as the difference between the extraordinary and the ordinary indices of refraction $\Delta n \equiv n_{e}-n_{o}$.


Fig. 7 LC cell structure
2. $90^{\circ}$ Twisted Nematic LC Cell

In the $90^{\circ}$ twisted nematic (TN) cell shown in Fig. 8, the LC director of the back surface is twisted $90^{\circ}$ with respect to the front surface. The front local director is set parallel to the transmission axis of the polarizer. An incident unpolarized light is converted into a linearly polarized light by the front polarizer.


Fig. $8 \quad 90^{\circ}$ TN LC cell

When a linearly polarized light traverses through a $90^{\circ} \mathrm{TN}$ cell, its polarization follows the twist of the LC directors (polarized light sees $\mathrm{n}_{\mathrm{e}}$ only) so that the output beam remains linearly polarized except for that its polarization axis is rotated by $90^{\circ}$ (it's called the polarizing rotary effect by $\mathrm{n}_{\mathrm{e}}$; similarly we can also find polarizing rotary effect by $\mathrm{n}_{\mathrm{o}}$ ). Thus, for a normally black (NB) mode using a $90^{\circ} \mathrm{TN}$ cell, the analyzer's (a second polarizer) transmission axis is set to be parallel to the polarizer's transmission axis, as shown in Fig. 9. However, when the applied voltage V across the LC cell exceeds a critical value $V_{c}$, the director of LC molecules tends to align along the direction of applied external electrical field which is in the direction of the propagation of light. Hence, the polarization guiding effect of the LC cell is gradually diminishing and the light leaks through the analyzer. Its electro-optical switching slope $\gamma$ is defined as $\left(\mathrm{V}_{90}-\mathrm{V}_{10}\right) / \mathrm{V}_{10}$, where $\mathrm{V}_{10}$ and $\mathrm{V}_{90}$ are the applied voltages enabling output light signal reaches up to $10 \%$ and $90 \%$ of its maximum light intensity, respectively.


Fig. 9 NB mode operation of a $90^{\circ} \mathrm{TN}$ cell

## II. Experiments and procedures

1. Setup a NB $90^{\circ} \mathrm{TN}$ LC mode between two polarizers with parallel transmission axes and apply 100 Hz square wave voltage using a function generator onto the ITO portions of two glass substrates and vary the applied voltage ( $\mathrm{V}_{\mathrm{rms}}$ ) from 0 to 7.2 Volts.

* In the crucial turning points, take more data if necessary.


## Question_B-(1) (5.0 points)

Measure, tabulate, and plot the electro-optical switching curve ( $\mathcal{J}$ vs. $\mathrm{V}_{\mathrm{rms}}$ curve) of the NB $90^{\circ}$ TN LC, and find its switching slope $\gamma$, where $\gamma$ is defined as $\left(\mathrm{V}_{90}-\mathrm{V}_{10}\right) / \mathrm{V}_{10}$.

Question B-(2) (2.5 points)
Determine the critical voltage $V_{c}$ of this NB $90^{\circ}$ TN LC cell. Show explicitly with graph how you determine the value $\mathrm{V}_{\mathrm{c}}$.
Hint:* When the external applied voltage exceeds the critical voltage, the light transmission increases rapidly and abruptly.

## Part C Optical Properties of Nematic Liquid Crystal : Electro-optical switching characteristic of parallel aligned LC cell

## I. Introduction

## Homogeneous Parallel-aligned LC Cell

For a parallel-aligned LC cell, the directors in the front and back substrates are parallel with each other, as shown in Fig. 10. When a linearly polarized light impinges on a parallel-aligned cell with its polarization parallel to the LC director (rubbing direction), a pure phase modulation is achieved because the light behaves only as an extraordinary ray.


Glass substrate (ITO+PI)
Fig. 10 Homogeneous parallel aligned LC

On the other hand, if a linearly polarized light is normally incident onto a parallel aligned cell but with its polarization making $\theta=45^{\circ}$ relative to the direction of the aligned LC directors (Fig. 11), then phase retardation occurs due to the different propagating speed of the extraordinary and ordinary rays in the LC medium. In this $\theta=45^{\circ}$ configuration, when the two polarizers are parallel, the normalized transmission of a parallel aligned LC cell is given by

$$
T_{\|}=\cos ^{2} \frac{\delta}{2}
$$

The phase retardation $\delta$ is expressed as

$$
\delta=2 \pi d \Delta n(V, \lambda) / \lambda
$$

where $d$ is the LC layer thickness, $\lambda$ is the wavelength of light in air, $V$ is the root mean square of applied AC voltage, and $\Delta n$, a function of $\lambda$ and $V$, is the LC birefringence. It should be also noted that, at $V=0, \Delta n\left(=n_{\mathrm{e}}-n_{\mathrm{o}}\right)$ has its maximum value, so does $\delta$. Also $\Delta n$ decreases as $V$ increases.

In the general case, we have

$$
T_{/ /}=1-\sin ^{2} 2 \theta \sin ^{2} \frac{\delta}{2}
$$

$$
T_{\perp}=\sin ^{2} 2 \theta \sin ^{2} \frac{\delta}{2}
$$

where // and $\perp$ represent that the transmission axis of analyzer is parallel and perpendicular to that of the polarizer, respectively.

## II. Experiments and procedures

1. Replace NB $90^{\circ} \mathrm{TN}$ LC cell with parallel-aligned LC cell.
2. Set up $\theta=45^{\circ}$ configuration at $V=0$ as shown in Fig. 11. Let the analyzer's transmission axis perpendicular to that of the polarizer, then rotate the parallel-aligned LC cell until the intensity of the transmitted light reaches the maximum value ( $T_{\perp}$ ). This procedure establishes the $\theta=45^{\circ}$ configuration. Take down $T_{\perp}$ value, then, measure the intensity of the transmitted light ( $T_{/ /}$) of the same LC cell at the analyzer's transmission axis parallel to that of the polarizer (also at $V=0$ ).


Fig. 11 Schematic diagram of experimental setup (The arrow L is the alignment direction.)

Question C-(1) (2.5 points)
Assume that the wavelength of laser light 650 nm , LC layer thickness $7.7 \mu \mathrm{~m}$, and approximate value of $\Delta \mathrm{n} \approx 0.25$ are known. From the experimental data $\mathrm{T}_{\perp}$ and $\mathrm{T}_{\|}$obtained above, calculate the accurate value of the phase retardation $\delta$ and accurate value of birefringence $\Delta \mathrm{n}$ of this LC cell at $\mathrm{V}=0$.
3. Similar to the above experiment (1), in the $\theta=45^{\circ}$ configuration, apply 100 Hz square wave voltage using a function generator onto the ITO portions of two glass substrates, vary the applied voltage ( $\mathrm{V}_{r m s}$ ) from 0 to 7 Volts and measure the electro-optical switching curve ( $\mathrm{T}_{\|}$) at the analyzer's transmission axis parallel to the polarizer's transmission axis. (Hint: Measuring the $T_{\perp}$ switching curve is helpful to increase the data accuracy of the above $\mathrm{T}_{\|}$measurement; the data of $\mathrm{T}_{\perp}$ are not needed in the following questions. )

* In the crucial turning points, take more data if necessary (especially in the range of 0.5-4.0 Volts).

Question_C-(2) (3.0 points)
Measure, tabulate, and plot the electro-optical switching curve for $\mathrm{T}_{\|}$of this parallel aligned LC cell in the $\theta=45^{\circ}$ configuration.

Question C-(3) (2.0 points)
From the electro-optical switching data, find the value of the external applied voltage $\mathrm{V}_{\pi}$. Hint: * $\mathrm{V}_{\pi}$ is the applied voltage which enables the phase retardation of this anisotropic LC cell become $\pi$ (or $180^{\circ}$ ).

* Remember that $\Delta \mathrm{n}$ is a function of applied voltage, and $\Delta \mathrm{n}$ decreases as V increases.
* Interpolation is probably needed when you determine the accurate value of this $\mathrm{V}_{\pi}$.


## Solution to Theoretical Question 1

## A Swing with a Falling Weight

## Part A

(a) Since the length of the string $L=s+R \theta$ is constant, its rate of change must be zero. Hence we have

$$
\begin{equation*}
\dot{s}+R \dot{\theta}=0 \tag{A1}
\end{equation*}
$$

(b) Relative to $O, Q$ moves on a circle of radius $R$ with angular velocity $\dot{\theta}$, so

$$
\begin{equation*}
\vec{v}_{Q}=R \dot{\theta} \hat{t}=-\dot{s} \hat{t} \tag{A2}
\end{equation*}
$$

(c) Refer to Fig. A1. Relative to $Q$, the displacement of $P$ in a time interval $\Delta t$ is $\Delta \vec{r}^{\prime}=(s \Delta \theta)(-\hat{r})+(\Delta s) \hat{t}=[(s \dot{\theta})(-\hat{r})+\dot{s} \hat{t}] \Delta t$. It follows

$$
\begin{equation*}
\bar{v}^{\prime}=-s \dot{\theta} \hat{r}+\dot{s} \hat{t} \tag{A3}
\end{equation*}
$$



Figure A1
(d) The velocity of the particle relative to $O$ is the sum of the two relative velocities given in Eqs. (A2) and (A3) so that

$$
\begin{equation*}
\bar{v}=\bar{v}^{\prime}+\vec{v}_{Q}=(-s \dot{\theta} \hat{r}+\dot{s} \hat{t})+R \dot{\theta} \hat{t}=-s \dot{\theta} \hat{r} \tag{A4}
\end{equation*}
$$

(e) Refer to Fig. A2. The ( $-\hat{t}$ )-component of the velocity change $\Delta \vec{v}$ is given by $(-\hat{t}) \cdot \Delta \vec{v}=v \Delta \theta=v \dot{\theta} \Delta t$. Therefore, the $\hat{t}$-component of the acceleration $\vec{a}=\Delta \vec{v} / \Delta t$ is given by $\hat{t} \cdot \hat{a}=-v \dot{\theta}$. Since the speed $v$ of the particle is $s \dot{\theta}$ according to Eq. (A4), we see that the $\hat{t}$-component of the particle's acceleration at $P$ is given by

$$
\begin{equation*}
\vec{a} \cdot \hat{t}=-v \dot{\theta}=-(s \dot{\theta}) \dot{\theta}=-s \dot{\theta}^{2} \tag{A5}
\end{equation*}
$$



Figure A2

Note that, from Fig. A2, the radial component of the acceleration may also be obtained as $\vec{a} \cdot \hat{r}=-d v / d t=-d(s \dot{\theta}) / d t$.
(f) Refer to Fig. A3. The gravitational potential energy of the particle is given by $U=-m g h$. It may be expressed in terms of $s$ and $\theta$ as

$$
\begin{equation*}
U(\theta)=-m g[R(1-\cos \theta)+s \sin \theta] \tag{A6}
\end{equation*}
$$



Figure A3
(g) At the lowest point of its trajectory, the particle's gravitational potential energy $U$ must assume its minimum value $U_{m}$. By differentiating Eq. (A6) with respect to $\theta$ and using Eq. (A1), the angle $\theta_{m}$ corresponding to the minimum gravitational energy can be obtained.

$$
\begin{aligned}
\frac{d U}{d \theta} & =-m g\left(R \sin \theta+\frac{d s}{d \theta} \sin \theta+s \cos \theta\right) \\
& =-m g[R \sin \theta+(-R) \sin \theta+s \cos \theta] \\
& =-m g s \cos \theta
\end{aligned}
$$

At $\theta=\theta_{m},\left.\frac{d U}{d \theta}\right|_{\theta_{m}}=0$. We have $\theta_{m}=\frac{\pi}{2}$. The lowest point of the particle's trajectory is shown in Fig. A4 where the length of the string segment of QP is $s=L^{-} \pi R / 2$.


Figure A4
From Fig. A4 or Eq. (A6), the minimum potential energy is then

$$
\begin{equation*}
U_{m}=U(\pi / 2)=-m g[R+L-(\pi R / 2)] \tag{A7}
\end{equation*}
$$

Initially, the total mechanical energy $E$ is 0 . Since $E$ is conserved, the speed $v_{m}$ of the particle at the lowest point of its trajectory must satisfy

$$
\begin{equation*}
E=0=\frac{1}{2} m v_{m}^{2}+U_{m} \tag{A8}
\end{equation*}
$$

From Eqs. (A7) and (A8), we obtain

$$
\begin{equation*}
v_{m}=\sqrt{-2 U_{m} / m}=\sqrt{2 g[R+(L-\pi R / 2)]} \tag{A9}
\end{equation*}
$$

## Part B

(h) From Eq. (A6), the total mechanical energy of the particle may be written as

$$
\begin{equation*}
E=0=\frac{1}{2} m v^{2}+U(\theta)=\frac{1}{2} m v^{2}-m g[R(1-\cos \theta)+s \sin \theta] \tag{B1}
\end{equation*}
$$

From Eq. (A4), the speed $v$ is equal to $s \dot{\theta}$. Therefore, Eq. (B1) implies

$$
\begin{equation*}
v^{2}=(s \dot{\theta})^{2}=2 g[R(1-\cos \theta)+s \sin \theta] \tag{B2}
\end{equation*}
$$

Let $T$ be the tension in the string. Then, as Fig. B1 shows, the $\hat{t}$-component of the net force on the particle is $-T+m g \sin \theta$. From Eq. (A5), the tangential acceleration of the particle is $\left(-s \dot{\theta}^{2}\right)$. Thus, by Newton's second law, we have

$$
\begin{equation*}
m\left(-s \dot{\theta}^{2}\right)=-T+m g \sin \theta \tag{B3}
\end{equation*}
$$



Figure B1

According to the last two equations, the tension may be expressed as

$$
\begin{align*}
T & =m\left(s \dot{\theta}^{2}+g \sin \theta\right)=\frac{m g}{s}[2 R(1-\cos \theta)+3 s \sin \theta] \\
& =\frac{2 m g R}{s}\left[\tan \frac{\theta}{2}-\frac{3}{2}\left(\theta-\frac{L}{R}\right)\right](\sin \theta)  \tag{B4}\\
& =\frac{2 m g R}{s}\left(y_{1}-y_{2}\right)(\sin \theta)
\end{align*}
$$

The functions $y_{1}=\tan (\theta / 2)$ and $y_{2}=3(\theta-L / R) / 2$ are plotted in Fig B2.


From Eq. (B4) and Fig. B2, we obtain the result shown in Table B1. The angle at which $. y_{2}=y_{1}$ is called $\theta_{S}\left(\pi<\theta_{s}<2 \pi\right)$ and is given by

$$
\begin{equation*}
\frac{3}{2}\left(\theta_{s}-\frac{L}{R}\right)=\tan \frac{\theta_{s}}{2} \tag{B5}
\end{equation*}
$$

or, equivalently, by

$$
\begin{equation*}
\frac{L}{R}=\theta_{s}-\frac{2}{3} \tan \frac{\theta_{s}}{2} \tag{B6}
\end{equation*}
$$

Since the ratio $L / R$ is known to be given by

$$
\begin{equation*}
\frac{L}{R}=\frac{9 \pi}{8}+\frac{2}{3} \cot \frac{\pi}{16}=\left(\pi+\frac{\pi}{8}\right)-\frac{2}{3} \tan \frac{1}{2}\left(\pi+\frac{\pi}{8}\right) \tag{B7}
\end{equation*}
$$

one can readily see from the last two equations that $\theta_{s}=9 \pi / 8$.
Table B1

|  | $\left(y_{1}-y_{2}\right)$ | $\sin \theta$ | tension $T$ |
| :---: | :---: | :---: | :---: |
| $0<\theta<\pi$ | positive | positive | positive |
| $\theta=\pi$ | $+\infty$ | 0 | positive |
| $\pi<\theta<\theta_{s}$ | negative | negative | positive |
| $\theta=\theta_{s}$ | zero | negative | zero |
| $\theta_{s}<\theta<2 \pi$ | positive | negative | negative |

Table B1 shows that the tension $T$ must be positive (or the string must be taut and straight) in the angular range $0<\theta<\theta_{s}$. Once $\theta$ reaches $\theta_{s}$, the tension $T$ becomes zero and the part of the string not in contact with the rod will not be straight afterwards. The shortest possible value $s_{\text {min }}$ for the length $s$ of the line segment $Q P$ therefore occurs at $\theta=\theta_{s}$ and is given by

$$
\begin{equation*}
s_{\min }=L-R \theta_{s}=R\left(\frac{9 \pi}{8}+\frac{2}{3} \cot \frac{\pi}{16}-\frac{9 \pi}{8}\right)=\frac{2 R}{3} \cot \frac{\pi}{16}=3.352 R \tag{B8}
\end{equation*}
$$

When $\theta=\theta_{s}$, we have $T=0$ and Eqs. (B2) and (B3) then leads to $v_{s}^{2}=-g s_{\min } \sin \theta_{s}$. Hence the speed $v_{s}$ is

$$
\begin{align*}
v_{s} & =\sqrt{-g s_{\min } \sin \theta_{s}}=\sqrt{\frac{2 g R}{3} \cot \frac{\pi}{16} \sin \frac{\pi}{8}}=\sqrt{\frac{4 g R}{3}} \cos \frac{\pi}{16}  \tag{B9}\\
& =1.133 \sqrt{g R}
\end{align*}
$$

(i) When $\theta \geq \theta_{s}$, the particle moves like a projectile under gravity. As shown in Fig. B3, it is projected with an initial speed $v_{s}$ from the position $P=\left(x_{s}, y_{s}\right)$ in a direction making an angle $\phi=\left(3 \pi / 2-\theta_{s}\right)$ with the $y$-axis.
The speed $v_{H}$ of the particle at the highest point of its parabolic trajectory is equal to the $y$-component of its initial velocity when projected. Thus,

$$
\begin{equation*}
v_{H}=v_{s} \sin \left(\theta_{s}-\pi\right)=\sqrt{\frac{4 g R}{3}} \cos \frac{\pi}{16} \sin \frac{\pi}{8}=0.4334 \sqrt{g R} \tag{B10}
\end{equation*}
$$

The horizontal distance $H$ traveled by the particle from point $P$ to the point of maximum height is

$$
\begin{equation*}
H=\frac{v_{s}^{2} \sin 2\left(\theta_{s}-\pi\right)}{2 g}=\frac{v_{s}^{2}}{2 g} \sin \frac{9 \pi}{4}=0.4535 R \tag{B11}
\end{equation*}
$$



The coordinates of the particle when $\theta=\theta_{s}$ are given by

$$
\begin{align*}
& x_{s}=R \cos \theta_{s}-s_{\min } \sin \theta_{s}=-R \cos \frac{\pi}{8}+s_{\min } \sin \frac{\pi}{8}=0.358 R  \tag{B12}\\
& y_{s}=R \sin \theta_{s}+s_{\min } \cos \theta_{s}=-R \sin \frac{\pi}{8}-s_{\min } \cos \frac{\pi}{8}=-3.478 R \tag{B13}
\end{align*}
$$

Evidently, we have $\left|y_{s}\right|>(R+H)$. Therefore the particle can indeed reach its maximum height without striking the surface of the rod.

## Part C

(j) Assume the weight is initially lower than $O$ by $h$ as shown in Fig. C1.


When the weight has fallen a distance $D$ and stopped, the law of conservation of total mechanical energy as applied to the particle-weight pair as a system leads to

$$
\begin{equation*}
-M g h=E^{\prime}-M g(h+D) \tag{C1}
\end{equation*}
$$

where $E^{\prime}$ is the total mechanical energy of the particle when the weight has stopped. It follows

$$
\begin{equation*}
E^{\prime}=M g D \tag{C2}
\end{equation*}
$$

Let $\Lambda$ be the total length of the string. Then, its value at $\theta=0$ must be the same as at any other angular displacement $\theta$. Thus we must have

$$
\begin{equation*}
\Lambda=L+\frac{\pi}{2} R+h=s+R\left(\theta+\frac{\pi}{2}\right)+(h+D) \tag{C3}
\end{equation*}
$$

Noting that $D=\alpha L$ and introducing $\ell=L-D$, we may write

$$
\begin{equation*}
\ell=L-D=(1-\alpha) L \tag{C4}
\end{equation*}
$$

From the last two equations, we obtain

$$
\begin{equation*}
s=L-D-R \theta=\ell-R \theta \tag{C5}
\end{equation*}
$$

After the weight has stopped, the total mechanical energy of the particle must be conserved. According to Eq. (C2), we now have, instead of Eq. (B1), the following equation:

$$
\begin{equation*}
E^{\prime}=M g D=\frac{1}{2} m v^{2}-m g[R(1-\cos \theta)+s \sin \theta] \tag{C6}
\end{equation*}
$$

The square of the particle's speed is accordingly given by

$$
\begin{equation*}
v^{2}=(s \dot{\theta})^{2}=\frac{2 M g D}{m}+2 g R\left[(1-\cos \theta)+\frac{s}{R} \sin \theta\right] \tag{C7}
\end{equation*}
$$

Since Eq. (B3) stills applies, the tension $T$ of the string is given by

$$
\begin{equation*}
-T+m g \sin \theta=m\left(-s \dot{\theta}^{2}\right) \tag{C8}
\end{equation*}
$$

From the last two equations, it follows

$$
\begin{align*}
T & =m\left(s \dot{\theta}^{2}+g \sin \theta\right) \\
& =\frac{m g}{s}\left[\frac{2 M}{m} D+2 R(1-\cos \theta)+3 s \sin \theta\right]  \tag{C9}\\
& =\frac{2 m g R}{s}\left[\frac{M D}{m R}+(1-\cos \theta)+\frac{3}{2}\left(\frac{\ell}{R}-\theta\right) \sin \theta\right]
\end{align*}
$$

where Eq. (C5) has been used to obtain the last equality.
We now introduce the function

$$
\begin{equation*}
f(\theta)=1-\cos \theta+\frac{3}{2}\left(\frac{\ell}{R}-\theta\right) \sin \theta \tag{C10}
\end{equation*}
$$

From the fact $\ell=(L-D) \gg R$, we may write

$$
\begin{equation*}
f(\theta) \approx 1+\frac{3}{2} \frac{\ell}{R} \sin \theta-\cos \theta=1+A \sin (\theta-\phi) \tag{C11}
\end{equation*}
$$

where we have introduced

$$
\begin{equation*}
A=\sqrt{1+\left(\frac{3}{2} \frac{\ell}{R}\right)^{2}} \quad, \quad \phi=\tan ^{-1}\left(\frac{2 R}{3 \ell}\right) \tag{C12}
\end{equation*}
$$

From Eq. (C11), the minimum value of $f(\theta)$ is seen to be given by

$$
\begin{equation*}
f_{\min }=1-A=1-\sqrt{1+\left(\frac{3}{2} \frac{\ell}{R}\right)^{2}} \tag{C13}
\end{equation*}
$$

Since the tension $T$ remains nonnegative as the particle swings around the rod, we have from Eq. (C9) the inequality

$$
\begin{equation*}
\frac{M D}{m R}+f_{\min }=\frac{M(L-\ell)}{m R}+1-\sqrt{1+\left(\frac{3 \ell}{2 R}\right)^{2}} \geq 0 \tag{C14}
\end{equation*}
$$

or

$$
\begin{equation*}
\left(\frac{M L}{m R}\right)+1 \geq\left(\frac{M \ell}{m R}\right)+\sqrt{1+\left(\frac{3 \ell}{2 R}\right)^{2}} \approx\left(\frac{M \ell}{m R}\right)+\left(\frac{3 \ell}{2 R}\right) \tag{C15}
\end{equation*}
$$

From Eq. (C4), Eq. (C15) may be written as

$$
\begin{equation*}
\left(\frac{M L}{m R}\right)+1 \geq\left(\frac{M L}{m R}+\frac{3 L}{2 R}\right)(1-\alpha) \tag{C16}
\end{equation*}
$$

Neglecting terms of the order $(R / L)$ or higher, the last inequality leads to

$$
\begin{equation*}
\alpha \geq 1-\frac{\left(\frac{M L}{m R}\right)+1}{\left(\frac{M L}{m R}+\frac{3 L}{2 R}\right)}=\frac{\frac{3 L}{2 R}-1}{\frac{M L}{m R}+\frac{3 L}{2 R}}=\frac{1-\frac{2 R}{3 L}}{\frac{2 M}{3 m}+1} \approx \frac{1}{1+\frac{2 M}{3 m}} \tag{C17}
\end{equation*}
$$

The critical value for the ratio $D / L$ is therefore

$$
\begin{equation*}
\alpha_{c}=\frac{1}{1+\frac{2 M}{3 m}} \tag{C18}
\end{equation*}
$$

## Marking Scheme

## Theoretical Question 1

A Swing with a Falling Weight

| Total Scores | Sub <br> Scores | Marking Scheme for Answers to the Problem |
| :---: | :---: | :---: |
| Part A <br> 4.3 pts. | (a) 0.5 | $\begin{aligned} & \text { Relation between } \dot{\theta} \text { and } \dot{s} . \quad(\dot{s}=-R \dot{\theta}) \\ & \\ & \\ & \\ & \\ & 0.2 \text { for } \dot{\theta} \propto \dot{s} . \end{aligned}$ |
|  | (b) $0.5$ | Velocity of $Q$ relative to $O . \quad\left(\vec{v}_{Q}=R \dot{\theta} \hat{t}\right)$ <br> $>0.2$ for magnitude $R \dot{\theta}$. <br> $>0.3$ for direction $\hat{t}$. |
|  | $\begin{aligned} & \hline \text { (c) } \\ & 0.7 \end{aligned}$ | Particle's velocity at $P$ relative to $Q .\left(\bar{v}^{\prime}=-s \dot{\theta} \hat{r}+\dot{s} \hat{t}\right)$ <br> $>0.2+0.1$ for magnitude and direction of $\hat{r}$-component. <br> $>0.3+0.1$ for magnitude and direction of $\hat{t}$-component. |
|  | $\begin{aligned} & \text { (d) } \\ & 0.7 \end{aligned}$ | Particle's velocity at $P$ relative to $O . \quad\left(\vec{v}=\vec{v}^{\prime}+\vec{v}_{Q}=-s \dot{\theta} \hat{r}\right)$ <br> $>0.3$ for vector addition of $\vec{v}^{\prime}$ and $\vec{v}_{Q}$. <br> $>0.2+0.2$ for magnitude and direction of $\vec{v}$. |
|  | (e) <br> 0.7 | ```\(\hat{t}\)-component of particle's acceleration at \(P\). 0.3 for relating \(\vec{a}\) or \(\vec{a} \cdot \hat{t}\) to the velocity in a way that implies \(\|\vec{a} \cdot \hat{t}|=v^{2} / s\). 0.4 for \(\vec{a} \cdot \hat{t}=-s \dot{\theta}^{2} \quad\) ( 0.1 for minus sign.)``` |
|  | $\begin{aligned} & \hline \text { (f) } \\ & 0.5 \end{aligned}$ | Potential energy $U$. <br> $>0.2$ for formula $U=-m g h$. <br> $>0.3$ for $h=R(1-\cos \theta)+s \sin \theta$ or $U$ as a function of $\theta, s$, and $R$. |
|  | $\begin{aligned} & \hline \text { (g) } \\ & 0.7 \end{aligned}$ | Speed at lowest point $v_{m}$. <br> 0.2 for lowest point at $\theta=\pi / 2$ or $U$ equals minimum $U_{m}$. <br> 0.2 for total mechanical energy $E=m v_{m}^{2} / 2+U_{m}=0$. <br> 0.3 for $v_{m}=\sqrt{-2 U_{m} / m}=\sqrt{2 g[R+(L-\pi R / 2)]}$. |
| Part B <br> 4.3 pts. | (h) 2.4 | Particle's speed $v_{s}$ when $\overline{Q P}$ is shortest. <br> 0.4 for tension $T$ becomes zero when $\overline{Q P}$ is shortest. <br> 0.3 for equation of motion $-T+m g \sin \theta=m\left(-s \dot{\theta}^{2}\right)$. <br> 0.3 for $E=0=m(s \dot{\theta})^{2} / 2-m g[R(1-\cos \theta)+s \sin \theta]$. <br> 0.4 for $\frac{3}{2}\left(\theta_{s}-\frac{L}{R}\right)=\tan \frac{\theta_{s}}{2}$. <br> 0.5 for $\theta_{s}=9 \pi / 8$. <br> $0.3+0.2$ for $v_{s}=\sqrt{4 g R / 3} \cos \pi / 16=1.133 \sqrt{g R}$ |


|  | (i) 1.9 | The speed $v_{H}$ of the particle at its highest point. <br> 0.4 for particle undergoes projectile motion when $\theta \geq \theta_{s}$. <br> 0.3 for angle of projection $\phi=\left(3 \pi / 2-\theta_{s}\right)$. <br> 0.3 for $v_{H}$ is the $y$-component of its velocity at $\theta=\theta_{s}$. <br> 0.4 for noting particle does not strike the surface of the rod. <br> $0.3+0.2$ for $v_{H}=\sqrt{4 g R / 3} \cos (\pi / 16) \sin (\pi / 8)=0.4334 \sqrt{g R} .$ |
| :---: | :---: | :---: |
| Part C 3.4 pts | (j) 3.4 | ```The critical value \(\alpha_{c}\) of the ratio \(D / L\). 0.4 for particle's energy \(E^{\prime}=M g D\) when the weight has stopped. 0.3 for \(s=L-D-R \theta\). 0.3 for \(E^{\prime}=M g D=m v^{2} / 2-m g[R(1-\cos \theta)+s \sin \theta]\). 0.3 for \(-T+m g \sin \theta=m\left(-s \dot{\theta}^{2}\right)\). 0.3 for concluding \(T\) must not be negative. 0.6 for an inequality leading to the determination of the range of \(D / L\). 0.6 for solving the inequality to give the range of \(\alpha=D / L\). 0.6 for \(\alpha_{c}=(1+2 M / 3 m)\).``` |

## Solution to Theoretical Question 2

## A Piezoelectric Crystal Resonator under an Alternating Voltage

## Part A

(a) Refer to Figure A1. The left face of the rod moves a distance $v \Delta t$ while the pressure wave travels a distance $u \Delta t$ with $u=\sqrt{Y / \rho}$. The strain at the left face is

$$
\begin{equation*}
S=\frac{\Delta \ell}{\ell}=\frac{-v \Delta t}{u \Delta t}=\frac{-v}{u} \tag{A1a}
\end{equation*}
$$

From Hooke's law, the pressure at the left face is

$$
\begin{equation*}
p=-Y S=Y \frac{v}{u}=\rho u v \tag{A1b}
\end{equation*}
$$

Figure A1

(b) The velocity $v$ is related to the displacement $\xi$ as in a simple harmonic motion (or a uniform circular motion, as shown in Figure A2) of angular frequency $\omega=k u$. Therefore, if $\xi(x, t)=\xi_{0} \sin k(x-u t)$, then

$$
\begin{equation*}
v(x, t)=-k u \xi_{0} \cos k(x-u t) \tag{A2}
\end{equation*}
$$

The strain and pressure are related to velocity as in Problem (a). Hence,

$$
\begin{align*}
S(x, t) & =-v(x, t) / u=k \xi_{0} \cos k(x-u t)  \tag{A3}\\
p(x, t) & =\rho u v(x, t)=-k \rho u^{2} \xi_{0} \cos k(x-u t)  \tag{A4}\\
& =-Y S(x, t)=-k Y \xi_{0} \cos k(x-u t)
\end{align*}
$$

Alternatively, the answers may be obtained by differentiations:

$$
\begin{aligned}
& v(x, t)=\frac{\Delta \xi}{\Delta t}=-k u \xi_{0} \cos k(x-u t), \\
& S(x, t)=\frac{\Delta \xi}{\Delta x}=k \xi_{0} \cos k(x-u t), \\
& p(x, t)=-Y \frac{\Delta \xi}{\Delta x}=-k Y \xi_{0} \cos k(x-u t) .
\end{aligned}
$$

Figure A2


## Part B

(c) Since the angular frequency $\omega$ and speed of propagation $u$ are given, the wavelength is given by $\lambda=2 \pi / k$ with $k=\omega / u$. The spatial variation of the displacement $\xi$ is therefore described by

$$
\begin{equation*}
g(x)=B_{1} \sin k\left(x-\frac{b}{2}\right)+B_{2} \cos k\left(x-\frac{b}{2}\right) \tag{B1}
\end{equation*}
$$

Since the centers of the electrodes are assumed to be stationary, $g(b / 2)=0$. This leads to $B_{2}=0$. Given that the maximum of $g(x)$ is 1 , we have $B_{1}= \pm 1$ and

$$
\begin{equation*}
g(x)= \pm \sin \frac{\omega}{u}\left(x-\frac{b}{2}\right) \tag{B2}
\end{equation*}
$$

Thus, the displacement is

$$
\begin{equation*}
\xi(x, t)= \pm 2 \xi_{0} \sin \frac{\omega}{u}\left(x-\frac{b}{2}\right) \cos \omega t \tag{B3}
\end{equation*}
$$

(d) Since the pressure $p$ (or stress $T$ ) must vanish at the end faces of the quartz slab (i.e., $x=0$ and $x=b$ ), the answer to this problem can be obtained, by analogy, from the resonant frequencies of sound waves in an open pipe of length $b$. However, given that the centers of the electrodes are stationary, all even harmonics of the fundamental tone must be excluded because they have antinodes, rather than nodes, of displacement at the bisection plane of the slab.

Since the fundamental tone has a wavelength $\lambda=2 b$, the fundamental frequency is given by $f_{1}=u /(2 b)$. The speed of propagation $u$ is given by

$$
\begin{equation*}
u=\sqrt{\frac{Y}{\rho}}=\sqrt{\frac{7.87 \times 10^{10}}{2.65 \times 10^{3}}}=5.45 \times 10^{3} \mathrm{~m} / \mathrm{s} \tag{B4}
\end{equation*}
$$

and, given that $b=1.00 \times 10^{-2} \mathrm{~m}$, the two lowest standing wave frequencies are

$$
\begin{equation*}
f_{1}=\frac{u}{2 b}=273(\mathrm{kHz}), \quad f_{3}=3 f_{1}=\frac{3 u}{2 b}=818(\mathrm{kHz}) \tag{B5}
\end{equation*}
$$

[Alternative solution to Problems (c) and (d)]:
A longitudinal standing wave in the quartz slab has a displacement node at $x=b / 2$. It may be regarded as consisting of two waves traveling in opposite directions. Thus, its displacement and velocity must have the following form

$$
\begin{align*}
\xi(x, t) & =\xi_{m}\left[\sin k\left(x-\frac{b}{2}-u t\right)+\sin k\left(x-\frac{b}{2}+u t\right)\right] \\
& =2 \xi_{m} \sin k\left(x-\frac{b}{2}\right) \cos \omega t  \tag{B6}\\
v(x, t) & =-k u \xi_{m}\left[\cos k\left(x-\frac{b}{2}-u t\right)-\cos k\left(x-\frac{b}{2}+u t\right)\right]  \tag{B7}\\
& =-2 \omega \xi_{m} \sin k\left(x-\frac{b}{2}\right) \sin \omega t
\end{align*}
$$

where $\omega=k u$ and the first and second factors in the square brackets represent waves
traveling along the $+x$ and $-x$ directions, respectively. Note that Eq. (B6) is identical to Eq. (B3) if we set $\xi_{m}= \pm \xi_{0}$.

For a wave traveling along the $-x$ direction, the velocity $v$ must be replaced by $-v$ in Eqs. (A1a) and (A1b) so that we have

$$
\begin{array}{ll}
S=\frac{-v}{u} \text { and } p=\rho u v & \text { (waves traveling along }+x \text { ) } \\
S=\frac{v}{u} \quad \text { and } p=-\rho u v & \text { (waves traveling along }-x \text { ) } \tag{B9}
\end{array}
$$

As in Problem (b), the strain and pressure are therefore given by

$$
\begin{align*}
S(x, t) & =-k \xi_{m}\left[-\cos k\left(x-\frac{b}{2}-u t\right)-\cos k\left(x-\frac{b}{2}+u t\right)\right]  \tag{B10}\\
& =2 k \xi_{m} \cos k\left(x-\frac{b}{2}\right) \cos \omega t \\
p(x, t) & =-\rho u \omega \xi_{m}\left[\cos k\left(x-\frac{b}{2}-u t\right)+\cos k\left(x-\frac{b}{2}+u t\right)\right]  \tag{B11}\\
& =-2 \rho u \omega \xi_{m} \cos k\left(x-\frac{b}{2}\right) \cos \omega t
\end{align*}
$$

Note that $v, S$, and $p$ may also be obtained by differentiating $\xi$ as in Problem (b).
The stress $T$ or pressure $p$ must be zero at both ends ( $x=0$ and $x=b$ ) of the slab at all times because they are free. From Eq. (B11), this is possible only if $\cos (\mathrm{kb} / 2)=0$ or

$$
\begin{equation*}
k b=\frac{\omega}{u} b=\frac{2 \pi f}{\lambda f} b=n \pi, \quad n=1,3,5, \cdots \tag{B12}
\end{equation*}
$$

In terms of wavelength $\lambda$, Eq. (B12) may be written as

$$
\begin{equation*}
\lambda=\frac{2 b}{n}, \quad n=1,3,5, \cdots . \tag{B13}
\end{equation*}
$$

The frequency is given by

$$
\begin{equation*}
f=\frac{u}{\lambda}=\frac{n u}{2 b}=\frac{n}{2 b} \sqrt{\frac{Y}{\rho}}, \quad n=1,3,5, \cdots . \tag{B14}
\end{equation*}
$$

This is identical with the results given in Eqs. (B4) and (B5).
(e) From Eqs. (5a) and (5b) in the Question, the piezoelectric effect leads to the equations

$$
\begin{align*}
& T=Y\left(S-d_{p} E\right)  \tag{B15}\\
& \sigma=Y d_{p} S+\varepsilon_{T}\left(1-Y \frac{d_{p}^{2}}{\varepsilon_{T}}\right) E \tag{B16}
\end{align*}
$$

Because $x=b / 2$ must be a node of displacement for any longitudinal standing wave in the slab, the displacement $\xi$ and strain $S$ must have the form given in Eqs. (B6) and (B10), i.e., with $\omega=k u$,

$$
\begin{equation*}
\xi(x, t)=\xi_{m} \sin k\left(x-\frac{b}{2}\right) \cos (\omega t+\phi) \tag{B17}
\end{equation*}
$$

$$
\begin{equation*}
S(x, t)=k \xi_{m} \cos k\left(x-\frac{b}{2}\right) \cos (\omega t+\phi) \tag{B18}
\end{equation*}
$$

where a phase constant $\phi$ is now included in the time-dependent factors.
By assumption, the electric field $E$ between the electrodes is uniform and depends only on time:

$$
\begin{equation*}
E(x, t)=\frac{V(t)}{h}=\frac{V_{m} \cos \omega t}{h} \tag{B19}
\end{equation*}
$$

Substituting Eqs. (B18) and (B19) into Eq. (B15), we have

$$
\begin{equation*}
T=Y\left[k \xi_{m} \cos k\left(x-\frac{b}{2}\right) \cos (\omega t+\phi)-\frac{d_{p}}{h} V_{m} \cos \omega t\right] \tag{B20}
\end{equation*}
$$

The stress $T$ must be zero at both ends ( $x=0$ and $x=b$ ) of the slab at all times because they are free. This is possible only if $\phi=0$ and

$$
\begin{equation*}
k \xi_{m} \cos \frac{k b}{2}=d_{p} \frac{V_{m}}{h} \tag{B21}
\end{equation*}
$$

Since $\phi=0$, Eqs. (B16), (B18), and (B19) imply that the surface charge density must have the same dependence on time $t$ and may be expressed as

$$
\begin{equation*}
\sigma(x, t)=\sigma(x) \cos \omega t \tag{B22}
\end{equation*}
$$

with the dependence on $x$ given by

$$
\begin{align*}
\sigma(x) & =Y d_{p} k \xi_{m} \cos k\left(x-\frac{b}{2}\right)+\varepsilon_{T}\left(1-Y \frac{d_{p}^{2}}{\varepsilon_{T}}\right) \frac{V_{m}}{h} \\
& =\left[Y \frac{d_{p}^{2}}{\cos \frac{k b}{2}} \cos k\left(x-\frac{b}{2}\right)+\varepsilon_{T}\left(1-Y \frac{d_{p}^{2}}{\varepsilon_{T}}\right)\right] \frac{V_{m}}{h} \tag{B23}
\end{align*}
$$

(f) At time $t$, the total surface charge $Q(t)$ on the lower electrode is obtained by integrating $\sigma(x, t)$ in Eq. (B22) over the surface of the electrode. The result is

$$
\begin{align*}
\frac{Q(t)}{V(t)} & =\frac{1}{V(t)} \int_{0}^{b} \sigma(x, t) w d x=\frac{1}{V_{m}} \int_{0}^{b} \sigma(x) w d x \\
& =\frac{w}{h} \int_{0}^{b}\left[Y \frac{d_{p}^{2}}{\cos \frac{k b}{2}} \cos k\left(x-\frac{b}{2}\right)+\varepsilon_{T}\left(1-Y \frac{d_{p}^{2}}{\varepsilon_{T}}\right)\right] d x  \tag{B24}\\
& =\left(\varepsilon_{T} \frac{b w}{h}\right)\left[Y \frac{d_{p}^{2}}{\varepsilon_{T}}\left(\frac{2}{k b} \tan \frac{k b}{2}\right)+\left(1-Y \frac{d_{p}^{2}}{\varepsilon_{T}}\right)\right] \\
& =C_{0}\left[\alpha^{2}\left(\frac{2}{k b} \tan \frac{k b}{2}\right)+\left(1-\alpha^{2}\right)\right]
\end{align*}
$$

where

$$
\begin{equation*}
C_{0}=\varepsilon_{T} \frac{b w}{h}, \quad \alpha^{2}=Y \frac{d_{p}^{2}}{\varepsilon_{T}}=\frac{(2.25)^{2} \times 10^{-2}}{1.27 \times 4.06}=9.82 \times 10^{-3} \tag{B25}
\end{equation*}
$$

(The constant $\alpha$ is called the electromechanical coupling coefficient.)

Note: The result $C_{0}=\varepsilon_{T} b w / h$ can readily be seen by considering the static limit $k=0$ of Eq. (5) in the Question. Since $\tan x \approx x$ when $x \ll 1$, we have

$$
\begin{equation*}
\lim _{k \rightarrow 0} Q(t) / V(t) \approx C_{0}\left[\alpha^{2}+\left(1-\alpha^{2}\right)\right]=C_{0} \tag{B26}
\end{equation*}
$$

Evidently, the constant $C_{0}$ is the capacitance of the parallel-plate capacitor formed by the electrodes (of area $b w$ ) with the quartz slab (of thickness $h$ and permittivity $\varepsilon_{T}$ ) serving as the dielectric medium. It is therefore given by $\varepsilon_{T} b w / h$.

## Marking Scheme

## Theoretical Question 2

A Piezoelectric Crystal Resonator under an Alternating Voltage

| $\begin{array}{\|l\|} \hline \text { Total } \\ \text { Scores } \end{array}$ | $\begin{gathered} \hline \text { Sub } \\ \text { Scores } \end{gathered}$ | Marking Scheme for Answers to the Problem |
| :---: | :---: | :---: |
| Part A <br> 4.0 pts. | (a) 1.6 | The strain $S$ and pressure $p$ on the left face.  <br> T 0.4 for $\|\Delta \ell\|=v \Delta t$ and $\ell=u \Delta t$.  <br> $>0.4$ for $S=-v / u$. $(0.1 \mathrm{fr}$ ign) <br> $>0.4$ for relating $p$ to $S$ as $p=-Y S$. (0.1 for sign) <br> $>0.4$ for $p=\rho u v$. (0.1 for sign) |
|  | (b) 2.4 | The velocity $v(x, t)$, strain $S(x, t)$, and pressure $p(x, t)$. <br> > $0.3 \times 3$ sinusoidal variation with correct phase constant. ( 0.2 for phase constant.) <br> $>0.3 \times 3$ for amplitude. <br> $>0.2 \times 3$ for dependence on $x$ and $t$ as ( $k x-k u t$ ). |
| Part B <br> 6.0 pts | (c) 1.2 | The function $g(x)$ for a standing wave of angular frequency $\omega$. <br> 0.4 for $g(b / 2)=0$. <br> $>0.3+0.1$ for $B_{1}= \pm 1$ ( 0.1 for both signs) <br> $>0.4$ for $B_{2}=0$ |
|  |  | The two lowest standing wave frequencies. <br> $>0.2$ for wavelength of fundamental tone $\lambda=2 b$. <br> > 0.2 for excluding even harmonics. <br> $>(0.3+0.1)$ for $f_{1}=u / 2 b=273 \mathrm{kHz}$. <br> (0.1 for value) <br> $>(0.3+0.1)$ for $f_{3}=3 u / 2 b=818 \mathrm{kHz}$. <br> (0.1 for value) |
|  |  | ```The surface charge density \(\sigma\) as a function of \(x\) and \(t\). \(0.1 \times 2\) for \(\xi\) and \(S\), each a separable function of \(x\) and \(t\). \(0.1 \times 2\) for \(\xi\) and \(S\), each depends on time as \(\cos \omega t\) with \(\phi=0\). 0.3 for spatial part \(\xi(x)=\xi_{m} \sin k(x-b / 2)\). 0.3 for spatial part \(S(x)=k \xi_{m} \cos k(x-b / 2)\). 0.3 for \(T(x)=\left[k \xi_{m} \cos k(x-b / 2)-d_{p} V_{m} / h\right] Y\). 0.3 for \(k \xi_{m} \cos (k b / 2)=d_{p} V_{m} / h\). 0.6 for \(D_{1}(0.3)\) and \(D_{2}(0.3)\) in \(\sigma(x)\).``` |
|  | (f) 1.4 | ```The constants \(C_{0}\) and \(\alpha^{2}\). \(>0.2\) for relation between \(\sigma\) and \(Q\) as \(Q(t)=\left(\int_{0}^{b} \sigma(x) w d x\right) \cos \omega t\). 0.3 for noting \(Q(t) / V(t) \approx C_{0}\) as \(k \rightarrow 0\). 0.4 for \(C_{0}=\varepsilon_{T} b w / h\). \(0.4+0.1\) for \(\alpha^{2}=Y d_{p}^{2} / \varepsilon_{T}=9.82 \times 10^{-3}\). (0.1 for value)``` |

## Solution toTheoretical Question 3

## Part A

## Neutrino Mass and Neutron Decay

(a) Let $\left(c^{2} E_{e}, c \vec{q}_{e}\right),\left(c^{2} E_{p}, c \vec{q}_{p}\right)$, and ( $\left.c^{2} E_{v}, c \vec{q}_{v}\right)$ be the energy-momentum 4-vectors of the electron, the proton, and the anti-neutrino, respectively, in the rest frame of the neutron. Notice that $E_{e}, E_{p}, E_{v}, \vec{q}_{e}, \vec{q}_{p}, \vec{q}_{v}$ are all in units of mass. The proton and the anti-neutrino may be considered as forming a system of total rest mass $M_{c}$, total energy $c^{2} E_{c}$, and total momentum $c \vec{q}_{c}$. Thus, we have

$$
\begin{equation*}
E_{c}=E_{p}+E_{v}, \quad \vec{q}_{c}=\vec{q}_{p}+\vec{q}_{v}, \quad M_{c}^{2}=E_{c}^{2}-q_{c}^{2} \tag{A1}
\end{equation*}
$$

Note that the magnitude of the vector $\vec{q}_{c}$ is denoted as $q_{c}$. The same convention also applies to all other vectors.

Since energy and momentum are conserved in the neutron decay, we have

$$
\begin{gather*}
E_{c}+E_{e}=m_{n}  \tag{A2}\\
\vec{q}_{c}=-\vec{q}_{e} \tag{A3}
\end{gather*}
$$

When squared, the last equation leads to the following equality

$$
\begin{equation*}
q_{c}^{2}=q_{e}^{2}=E_{e}^{2}-m_{e}^{2} \tag{A4}
\end{equation*}
$$

From Eq. (A4) and the third equality of Eq. (A1), we obtain

$$
\begin{equation*}
E_{c}^{2}-M_{c}^{2}=E_{e}^{2}-m_{e}^{2} \tag{A5}
\end{equation*}
$$

With its second and third terms moved to the other side of the equality, Eq. (A5) may be divided by Eq. (A2) to give

$$
\begin{equation*}
E_{c}-E_{e}=\frac{1}{m_{n}}\left(M_{c}^{2}-m_{e}^{2}\right) \tag{A6}
\end{equation*}
$$

As a system of coupled linear equations, Eqs. (A2) and (A6) may be solved to give

$$
\begin{align*}
& E_{c}=\frac{1}{2 m_{n}}\left(m_{n}^{2}-m_{e}^{2}+M_{c}^{2}\right)  \tag{A7}\\
& E_{e}=\frac{1}{2 m_{n}}\left(m_{n}^{2}+m_{e}^{2}-M_{c}^{2}\right) \tag{A8}
\end{align*}
$$

Using Eq. (A8), the last equality in Eq. (A4) may be rewritten as

$$
\begin{align*}
q_{e} & =\frac{1}{2 m_{n}} \sqrt{\left(m_{n}^{2}+m_{e}^{2}-M_{c}^{2}\right)^{2}-\left(2 m_{n} m_{e}\right)^{2}} \\
& =\frac{1}{2 m_{n}} \sqrt{\left(m_{n}+m_{e}+M_{c}\right)\left(m_{n}+m_{e}-M_{c}\right)\left(m_{n}-m_{e}+M_{c}\right)\left(m_{n}-m_{e}-M_{c}\right)} \tag{A9}
\end{align*}
$$

Eq. (A8) shows that a maximum of $E_{e}$ corresponds to a minimum of $M_{c}^{2}$. Now the rest mass $M_{c}$ is the total energy of the proton and anti-neutrino pair in their center of mass (or momentum) frame so that it achieves the minimum

$$
\begin{equation*}
\left(M_{c}\right)_{\min }=M=m_{p}+m_{v} \tag{A10}
\end{equation*}
$$

when the proton and the anti-neutrino are both at rest in the center of mass frame. Hence, from Eqs. (A8) and (A10), the maximum energy of the electron $E=c^{2} E_{e}$ is

$$
\begin{equation*}
E_{\max }=\frac{c^{2}}{2 m_{n}}\left[m_{n}^{2}+m_{e}^{2}-\left(m_{p}+m_{v}\right)^{2}\right] \approx 1.292569 \mathrm{MeV} \approx 1.29 \mathrm{MeV} \tag{A11}
\end{equation*}
$$

When Eq. (A10) holds, the proton and the anti-neutrino move with the same velocity $v_{m}$ of the center of mass and we have

$$
\begin{equation*}
\frac{v_{m}}{c}=\left(\frac{q_{v}}{E_{v}}\right)_{E=E_{\max }}=\left(\frac{q_{p}}{E_{p}}\right)_{E=E_{\max }}=\left(\frac{q_{c}}{E_{c}}\right)_{E=E_{\max }}=\left(\frac{q_{e}}{E_{c}}\right)_{M_{c}=m_{p}+m_{v}} \tag{A12}
\end{equation*}
$$

where the last equality follows from Eq. (A3). By Eqs. (A7) and (A9), the last expression in Eq. (A12) may be used to obtain the speed of the anti-neutrino when $E=E_{\text {max. }}$. Thus, with $M=m_{p}+m_{v}$, we have

$$
\begin{align*}
\frac{v_{m}}{c} & =\frac{\sqrt{\left(m_{n}+m_{e}+M\right)\left(m_{n}+m_{e}-M\right)\left(m_{n}-m_{e}+M\right)\left(m_{n}-m_{e}-M\right)}}{m_{n}^{2}-m_{e}^{2}+M^{2}}  \tag{A13}\\
& \approx 0.00126538 \approx 0.00127
\end{align*}
$$

## [Alternative Solution]

Assume that, in the rest frame of the neutron, the electron comes out with momentum $c \vec{q}_{e}$ and energy $c^{2} E_{e}$, the proton with $c \vec{q}_{p}$ and $c^{2} E_{p}$, and the anti-neutrino with $c \vec{q}_{v}$ and $c^{2} E_{v}$. With the magnitude of vector $\vec{q}_{\alpha}$ denoted by the symbol $q_{\alpha}$, we have

$$
\begin{equation*}
E_{p}^{2}=m_{p}^{2}+q_{p}^{2}, \quad E_{v}^{2}=m_{v}^{2}+q_{v}^{2}, \quad E_{e}^{2}=m_{e}^{2}+q_{e}^{2} \tag{1A}
\end{equation*}
$$

Conservation of energy and momentum in the neutron decay leads to

$$
\begin{gather*}
E_{p}+E_{v}=m_{n}-E_{e}  \tag{2A}\\
\vec{q}_{p}+\vec{q}_{v}=-\vec{q}_{e} \tag{3A}
\end{gather*}
$$

When squared, the last two equations lead to

$$
\begin{align*}
& E_{p}^{2}+E_{v}^{2}+2 E_{p} E_{v}=\left(m_{n}-E_{e}\right)^{2}  \tag{4A}\\
& q_{p}^{2}+q_{v}^{2}+2 \vec{q}_{p} \cdot \vec{q}_{v}=q_{e}^{2}=E_{e}^{2}-m_{e}^{2} \tag{5A}
\end{align*}
$$

Subtracting Eq. (5A) from Eq. (4A) and making use of Eq. (1A) then gives

$$
\begin{equation*}
m_{p}^{2}+m_{v}^{2}+2\left(E_{p} E_{v}-\vec{q}_{p} \cdot \vec{q}_{v}\right)=m_{n}^{2}+m_{e}^{2}-2 m_{n} E_{e} \tag{6A}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
2 m_{n} E_{e}=m_{n}^{2}+m_{e}^{2}-m_{p}^{2}-m_{v}^{2}-2\left(E_{p} E_{v}-\vec{q}_{p} \cdot \vec{q}_{v}\right) \tag{7A}
\end{equation*}
$$

If $\theta$ is the angle between $\vec{q}_{p}$ and $\vec{q}_{v}$, we have $\vec{q}_{p} \cdot \vec{q}_{v}=q_{p} q_{v} \cos \theta \leq q_{p} q_{v}$ so that Eq. (7A) leads to the relation

$$
\begin{equation*}
2 m_{n} E_{e} \leq m_{n}^{2}+m_{e}^{2}-m_{p}^{2}-m_{v}^{2}-2\left(E_{p} E_{v}-q_{p} q_{v}\right) \tag{8A}
\end{equation*}
$$

Note that the equality in Eq. (8A) holds only if $\theta=0$, i.e., the energy of the electron $c^{2} E_{e}$ takes on its maximum value only when the anti-neutrino and the proton move in the same direction.

Let the speeds of the proton and the anti-neutrino in the rest frame of the neutron be $c \beta_{p}$ and $c \beta_{v}$, respectively. We then have $q_{p}=\beta_{p} E_{p}$ and $q_{v}=\beta_{v} E_{v}$. As shown in Fig. A1, we introduce the angle $\phi_{v}\left(0 \leq \phi_{v}<\pi / 2\right)$ for the antineutrino by

$$
\begin{equation*}
q_{v}=m_{v} \tan \phi_{v}, \quad E_{v}=\sqrt{m_{v}^{2}+q_{v}^{2}}=m_{v} \sec \phi_{v}, \quad \beta_{v}=q_{v} / E_{v}=\sin \phi_{v} \tag{9A}
\end{equation*}
$$



Figure A1

Similarly, for the proton, we write, with $0 \leq \phi_{p}<\pi / 2$,

$$
\begin{equation*}
q_{p}=m_{p} \tan \phi_{p}, \quad E_{p}=\sqrt{m_{p}^{2}+q_{p}^{2}}=m_{p} \sec \phi_{p}, \quad \beta_{p}=q_{p} / E_{p}=\sin \phi_{p} \tag{10A}
\end{equation*}
$$

Eq. (8A) may then be expressed as

$$
\begin{equation*}
2 m_{n} E_{e} \leq m_{n}^{2}+m_{e}^{2}-m_{p}^{2}-m_{v}^{2}-2 m_{p} m_{v}\left(\frac{1-\sin \phi_{p} \sin \phi_{v}}{\cos \phi_{p} \cos \phi_{v}}\right) \tag{11A}
\end{equation*}
$$

The factor in parentheses at the end of the last equation may be expressed as

$$
\begin{equation*}
\frac{1-\sin \phi_{p} \sin \phi_{v}}{\cos \phi_{p} \cos \phi_{v}}=\frac{1-\sin \phi_{p} \sin \phi_{v}-\cos \phi_{p} \cos \phi_{v}}{\cos \phi_{p} \cos \phi_{v}}+1=\frac{1-\cos \left(\phi_{p}-\phi_{v}\right)}{\cos \phi_{p} \cos \phi_{v}}+1 \geq 1 \tag{12A}
\end{equation*}
$$

and clearly assumes its minimum possible value of 1 when $\phi_{p}=\phi_{v}$, i.e., when the anti-neutrino and the proton move with the same velocity so that $\beta_{p}=\beta_{v}$. Thus, it follows from Eq. (11A) that the maximum value of $E_{e}$ is

$$
\begin{align*}
\left(E_{e}\right)_{\max } & =\frac{1}{2 m_{n}}\left(m_{n}^{2}+m_{e}^{2}-m_{p}^{2}-m_{v}^{2}-2 m_{p} m_{v}\right)  \tag{13A}\\
& =\frac{1}{2 m_{n}}\left[m_{n}^{2}+m_{e}^{2}-\left(m_{p}+m_{v}\right)^{2}\right]
\end{align*}
$$

and the maximum energy of the electron $E=c^{2} E_{e}$ is

$$
\begin{equation*}
E_{\max }=c^{2}\left(E_{e}\right)_{\max } \approx 1.292569 \mathrm{MeV} \approx 1.29 \mathrm{MeV} \tag{14A}
\end{equation*}
$$

When the anti-neutrino and the proton move with the same velocity, we have, from Eqs. (9A), (10A), (2A) , (3A), and (1A), the result

$$
\begin{equation*}
\beta_{v}=\beta_{p}=\frac{q_{p}}{E_{p}}=\frac{q_{v}}{E_{v}}=\frac{q_{p}+q_{v}}{E_{p}+E_{v}}=\frac{q_{e}}{m_{n}-E_{e}}=\frac{\sqrt{E_{e}^{2}-m_{e}^{2}}}{m_{n}-E_{e}} \tag{15A}
\end{equation*}
$$

Substituting the result of Eq. (13A) into the last equation, the speed $v_{m}$ of the anti-neutrino when the electron attains its maximum value $E_{\text {max }}$ is, with $M=m_{p}+m_{v}$, given by

$$
\begin{align*}
\frac{v_{m}}{c} & =\left(\beta_{v}\right)_{\max E_{e}}=\frac{\sqrt{\left(E_{e}\right)_{\max }^{2}-m_{e}^{2}}}{m_{n}-\left(E_{e}\right)_{\max }}=\frac{\sqrt{\left(m_{n}^{2}+m_{e}^{2}-M^{2}\right)^{2}-4 m_{n}^{2} m_{e}^{2}}}{2 m_{n}^{2}-\left(m_{n}^{2}+m_{e}^{2}-M^{2}\right)} \\
& =\frac{\sqrt{\left(m_{n}+m_{e}+M\right)\left(m_{n}+m_{e}-M\right)\left(m_{n}-m_{e}+M\right)\left(m_{n}-m_{e}-M\right)}}{m_{n}^{2}-m_{e}^{2}+M^{2}}  \tag{16A}\\
& \approx 0.00126538 \approx 0.00127
\end{align*}
$$

## Part B

## Light Levitation

(b) Refer to Fig. B1. Refraction of light at the spherical surface obeys Snell's law and leads to

$$
\begin{equation*}
n \sin \theta_{i}=\sin \theta_{t} \tag{B1}
\end{equation*}
$$

Neglecting terms of the order $(\delta / R)^{3}$ or higher in sine functions, Eq. (B1) becomes

$$
\begin{equation*}
n \theta_{i} \approx \theta_{t} \tag{B2}
\end{equation*}
$$

For the triangle $\triangle F A C$ in Fig. B1, we have

$$
\begin{equation*}
\beta=\theta_{t}-\theta_{i} \approx n \theta_{i}-\theta_{i}=(n-1) \theta_{i} \tag{B3}
\end{equation*}
$$

Let $f_{0}$ be the frequency of the incident light. If $n_{p}$ is the number of photons incident on the plane surface per unit area per unit time, then the total number of photons incident on the plane surface per unit time is $n_{p} \pi \delta^{2}$. The total power $P$ of photons incident on the plane surface is $\left(n_{p} \pi \delta^{2}\right)\left(h f_{0}\right)$, with $h$ being Planck's constant. Hence,

$$
\begin{equation*}
n_{p}=\frac{P}{\pi \delta^{2} h f_{0}} \tag{B4}
\end{equation*}
$$

The number of photons incident on an annular disk of inner radius $r$ and outer radius $r+d r$ on the plane surface per unit time is $n_{p}(2 \pi r d r)$, where $r=R \tan \theta_{i} \approx R \theta_{i}$. Therefore,


Fig. B1

$$
\begin{equation*}
n_{p}(2 \pi r d r) \approx n_{p}\left(2 \pi R^{2}\right) \theta_{i} d \theta_{i} \tag{B5}
\end{equation*}
$$

The $z$-component of the momentum carried away per unit time by these photons when
refracted at the spherical surface is

$$
\begin{align*}
d F_{z} & =n_{p} \frac{h f_{o}}{c}(2 \pi r d r) \cos \beta \approx n_{p} \frac{h f_{0}}{c}\left(2 \pi R^{2}\right)\left(1-\frac{\beta^{2}}{2}\right) \theta_{i} d \theta_{i} \\
& \approx n_{p} \frac{h f_{0}}{c}\left(2 \pi R^{2}\right)\left[\theta_{i}-\frac{(n-1)^{2}}{2} \theta_{i}^{3}\right] d \theta_{i} \tag{B6}
\end{align*}
$$

so that the $z$-component of the total momentum carried away per unit time is

$$
\begin{align*}
F_{z} & =2 \pi R^{2} n_{p}\left(\frac{h f_{0}}{c}\right) \int_{0}^{\theta_{i m}}\left[\theta_{i}-\frac{(n-1)^{2}}{2} \theta_{i}^{3}\right] d \theta_{i}  \tag{B7}\\
& =\pi R^{2} n_{p}\left(\frac{h f_{0}}{c}\right) \theta_{i m}^{2}\left[1-\frac{(n-1)^{2}}{4} \theta_{i m}^{2}\right]
\end{align*}
$$

where $\tan \theta_{\text {im }}=\frac{\delta}{R} \approx \theta_{\text {im }}$. Therefore, by the result of Eq. (B5), we have

$$
\begin{equation*}
F_{z}=\frac{\pi R^{2} P}{\pi \delta^{2} h f_{0}}\left(\frac{h f_{0}}{c}\right) \frac{\delta^{2}}{R^{2}}\left[1-\frac{(n-1)^{2} \delta^{2}}{4 R^{2}}\right]=\frac{P}{c}\left[1-\frac{(n-1)^{2} \delta^{2}}{4 R^{2}}\right] \tag{B8}
\end{equation*}
$$

The force of optical levitation is equal to the sum of the $z$-components of the forces exerted by the incident and refracted lights on the glass hemisphere and is given by

$$
\begin{equation*}
\frac{P}{c}+\left(-F_{z}\right)=\frac{P}{c}-\frac{P}{c}\left[1-\frac{(n-1)^{2} \delta^{2}}{4 R^{2}}\right]=\frac{(n-1)^{2} \delta^{2}}{4 R^{2}} \frac{P}{c} \tag{B9}
\end{equation*}
$$

Equating this to the weight $m g$ of the glass hemisphere, we obtain the minimum laser power required to levitate the hemisphere as

$$
\begin{equation*}
P=\frac{4 m g c R^{2}}{(n-1)^{2} \delta^{2}} \tag{B10}
\end{equation*}
$$

# Marking Scheme 

## Theoretical Question 3

Neutrino Mass and Neutron Decay

| Total Scores | $\begin{gathered} \hline \text { Sub } \\ \text { Scores } \end{gathered}$ | Marking Scheme for Answers to the Problem |
| :---: | :---: | :---: |
| Part A | (a) | The maximum energy of the electron and the corresponding speed of the anti-neutrino. |
| 4.0 pts. | 4.0 | 0.5 use energy-momentum conservation and can convert it into equations. <br> 0.5 obtain an expression for $E_{e}$ that allows the determination of its maximum value. <br> (0.5+0.2) for concluding that proton and anti-neutrino must move with the same velocity when $E_{e}$ is maximum. ( 0.2 for the same direction) <br> 0.6 for establishing the minimum value of $\left(E_{p} E_{v}-\vec{q}_{p} \cdot \vec{q}_{v}\right)$ to be $m_{p} m_{v}$ or a conclusion equivalent to it. <br> (0.5 +0.1 ) for expression and value of $E_{\text {max }}$. <br> 0.5 for concluding $\beta_{v}=\sqrt{E_{e}^{2}-m_{e}^{2}} /\left(m_{n}-E_{e}\right)$. <br> ( $0.5+0.1$ ) for expression and value of $v_{m} / c$. |

## Light Levitation

| Part B 4.0 pts | (b) 4.0 | Laser power needed to balance the weight of the glass hemisphere. <br> 0.3 for law of refraction $n \sin \theta_{i}=\sin \theta_{t}$. <br> 0.3 for making the linear approximation $n \theta_{i} \approx \theta_{t}$. <br> 0.4 for relation between angles of deviation and incidence. <br> 0.3 for photon energy $\varepsilon=h \nu$. <br> 0.3 for photon momentum $p=\varepsilon / c$. <br> 0.3 for momentum of incident photons per unit time $=P / c$. <br> 0.6 for momentum of photons refracted per unit time as a function of the angle of incidence. <br> 0.4 for total momentum of photons refracted per unit time $=$ $\left[1-(n-1)^{2} \delta^{2} /\left(4 R^{2}\right)\right] P / c$. <br> 0.4 for force of levitation = sum of forces exerted by incident and refracted photons. <br> 0.4 for force of levitation $=(n-1)^{2} \delta^{2} P /\left(4 c R^{2}\right)$. <br> 0.3 for the needed laser power $P=4 m g c R^{2} /(n-1)^{2} \delta^{2}$. |
| :---: | :---: | :---: |

## Solutions to Experimental Problems

## Part A: Optical Properties of Laser Diode

Question A-(1) (Total 1.5 point)
Measure, tabulate, and plot the $\mathcal{J}$ vs. $I$ curve.
a. Data ( 0.3 pts.) : Proper data table marked with variables and units.

Table A-(1): Data for $\mathcal{I}$ vs. $I$.

| $I(\mathrm{~mA})$ | 9.2 | 15.2 | 19.5 | 21.6 | 22.2 | 22.7 | 23.0 | 23.4 | 23.8 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{J}(\mathrm{~V})$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.05 | 0.06 | 0.09 | 0.12 | 0.30 |
| $I(\mathrm{~mA})$ | 24.2 | 24.6 | 25.0 | 25.4 | 25.8 | 26.2 | 26.6 | 27.0 | 27.4 |
| $\mathcal{J}(\mathrm{~V})$ | 0.66 | 1.02 | 1.41 | 1.88 | 2.23 | 2.64 | 3.04 | 3.36 | 3.78 |
| $I(\mathrm{~mA})$ | 27.8 | 28.2 | 28.6 | 29.0 | 29.4 | 29.8 | 30.2 | 30.5 | 31.0 |
| $\mathcal{J}(\mathrm{~V})$ | 4.12 | 4.48 | 4.79 | 5.13 | 5.44 | 5.72 | 6.05 | 6.25 | 6.55 |
| $I(\mathrm{~mA})$ | 31.4 | 31.8 | 32.2 | 32.6 | 33.0 | 33.4 | 33.8 | 34.2 | 34.6 |
| $\mathcal{J}(\mathrm{~V})$ | 6.75 | 6.99 | 7.22 | 7.40 | 7.60 | 7.78 | 7.93 | 8.07 | 8.14 |
| $I(\mathrm{~mA})$ | 35.0 | 35.5 | 36.0 | 36.5 | 37.0 | 37.6 | 38.0 | 38.6 |  |
| $\mathcal{J}(\mathrm{~V})$ | 8.18 | 8.20 | 8.22 | 8.24 | 8.24 | 8.25 | 8.26 | 8.27 |  |

Current error : $\pm 0.1 \mathrm{~mA}$; Voltage error : $\pm 0.01 \mathrm{~V}$
b. Plotting ( 0.3 pts.): Proper sizes of scales, and units for abscissa and ordinate that bear relation to the accuracy and range of the experiment.
c. Curve ( 0.9 pts.): Proper data and adequate line shape

- As shown in Fig. A-1. Start $\sim 0 \rightarrow$ Threshold $\rightarrow$ Linear $\rightarrow$ Saturate.


Fig. A-1 Graph of light intensity $\mathcal{J}$ versus current $I$

## Question A-(2) ( Total 3.5 points)

Estimate the maximum current $I_{m}$ with uncertainty in the linear region of the $\mathcal{J}-I$. Mark the linear region on the $\mathcal{J}-I$ curve figure by using arrows $(\downarrow)$ and determine the threshold current $I_{t h}$ with detailed error analysis.
a. Linear region marking ( 0.5 pts.) in Fig. A-1.
b. Least-square method or eye-balling with ruler and error analysis (1.5 pt.)

| Least-square fitting | Eye-balling with ruler |
| :--- | :--- |
| Error bar in graph $0.0 \mathrm{x} \mathrm{mA}(0.5 \mathrm{pts})$ | Error bar in graph $0 . \mathrm{x} \mathrm{mA}(0.5 \mathrm{pts})$ |
| Least-square method $(0.5 \mathrm{pts})$ | Expanded scale graph $(0.5 \mathrm{pts})$ |
| Error analysis $(0.5 \mathrm{pts})$ | draw three lines for error analysis $(0.5 \mathrm{pts})$ |

c. $I_{m} \pm \Delta I_{m}$ ( 0.5 pts.): Adequate value of $\mathrm{I}_{\mathrm{m}}$ ( 0.3 pts.) and error $\left( \pm \Delta I_{m}\right.$ ) ( 0.2 pts.) from the linear region of $\mathcal{J}$-I curve.
d. Adequate value of $\mathrm{I}_{\mathrm{th}}$ with error (1.0 pts.)
$I_{t h}=(21 \sim 26) \pm$ ( 0.01 or 0.2 for single value) mA
Adequate value of $I_{t h}\left(0.5\right.$ pts.) and error $\left( \pm \Delta I_{t h}\right)(0.5 \mathrm{pts}$.)


Fig. A-2 Straight lines and extrapolations

## Appendix :

## OA1-1

- Least-Square Method :
$I=m \mathcal{J}+b \quad \rightarrow \quad b=I_{\mathrm{th}}$
For $y=m x+b$

|  | $y: I(\mathrm{~mA})$ | $x: \mathcal{J}$ | $x y$ | $x^{2}$ | $y(x)=m x+b$ | $(y-y(x))^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 23.8 | 0.30 | 7.14 | 0.090 | 23.7937 | 3.969E-05 |
| 2 | 24.2 | 0.66 | 15.972 | 0.4356 | 24.17134 | 0.000821 |
| 3 | 24.6 | 1.02 | 25.092 | 1.0404 | 24.54898 | 0.00260 |
| 4 | 25.0 | 1.41 | 35.25 | 1.9881 | 24.95809 | 0.00176 |
| 5 | 25.4 | 1.88 | 47.752 | 3.5344 | 25.45112 | 0.00261 |
| 6 | 25.8 | 2.23 | 57.534 | 4.9729 | 25.81827 | 0.000334 |
| 7 | 26.2 | 2.64 | 69.168 | 6.9696 | 26.24836 | 0.00234 |
| 8 | 26.6 | 3.04 | 80.864 | 9.2416 | 26.66796 | 0.00462 |
| 9 | 27.0 | 3.36 | 90.72 | 11.2896 | 27.00364 | $1.325 \mathrm{E}-05$ |
| 10 | 27.4 | 3.78 | 103.572 | 14.2884 | 27.44422 | 0.00196 |
| 11 | 27.8 | 4.12 | 114.536 | 16.9744 | 27.80088 | $7.744 \mathrm{E}-07$ |
| 12 | 28.2 | 4.48 | 126.336 | 20.0704 | 28.17852 | 0.000461 |
| 13 | 28.6 | 4.79 | 136.994 | 22.9441 | 28.50371 | 0.00927 |
|  | $\begin{gathered} \Sigma y= \\ 340.6 \end{gathered}$ | $\begin{array}{r} \Sigma x= \\ 33.71 \end{array}$ | $\begin{gathered} \Sigma x y= \\ 910.93 \end{gathered}$ | $\begin{array}{r} \Sigma x^{2}= \\ 113.840 \end{array}$ |  | $\begin{array}{r} \Sigma(y-y(x))^{2}= \\ 0.0268 \end{array}$ |

$\Delta=N \Sigma x^{2}-(\Sigma x)^{2}=13(113.840)-(33.71)^{2}=343.556$
$m=\frac{1}{\Delta}(N \Sigma x y-\Sigma x \Sigma y)=\frac{13(910.93)-(33.71)(340.6)}{343.556}=1.049$
$b=\frac{1}{\Delta}\left(\Sigma x^{2} \Sigma y-\Sigma x \Sigma x y\right)=\frac{(113.840)(340.6)-(33.71)(910.93)}{343.556}=23.479$
$\sigma_{y}=\sqrt{\frac{\Sigma(y-y(x))^{2}}{N-2}}=\sqrt{\frac{0.0268}{13-2}}=0.049$
$\sigma=\sqrt{\left(\sigma_{y}\right)^{2}+\left(\frac{d y}{d x} \sigma_{x}\right)^{2}}=\sqrt{(0.049)^{2}+(1.049 \times 0.005)^{2}}=0.049$
$\sigma_{m}=\sqrt{\frac{N \sigma^{2}}{\Delta}}=\sqrt{\frac{13 \times 0.049^{2}}{343.556}}=0.0095$
$\sigma_{b}=\sqrt{\frac{\sigma^{2}}{\Delta} \Sigma x^{2}}=0.049 \times \sqrt{\frac{113.840}{343.556}}=0.028$
$I_{t h}=23.48 \pm 0.03 \mathrm{~mA}$
(ㅇ) A1-2

- Eye-balling Method :
$I=m \mathcal{J}+b \quad \rightarrow \quad b=I_{\mathrm{th}}$
For $y=m x+b$
Line 1: $y=1.00 x+23.66$
Line 2: $y=1.05 x+23.48$
Line3: $y=1.13 x+23.31$
$I_{\mathrm{th}}(\mathrm{av})=$.
$I_{\mathrm{th}}(\mathrm{std})=$.
$I_{t h}=23.5 \pm 0.2 \mathrm{~mA}$

Part B: Optical Properties of Nematic Liquid Crystal Electro-optical switching characteristic of $90^{\circ} \mathrm{TN}$ LC cell

Question_B-(1) (5.0 points)
Measure, tabulate, and plot the electro-optical switching curve ( $\mathcal{J}$ vs. $\mathrm{V}_{\text {rms }}$ curve) of the NB $90^{\circ}$ TN LC, and find its switching slope $\gamma$, where $\gamma$ is defined as $\left(\mathrm{V}_{90}-\mathrm{V}_{10}\right) / \mathrm{V}_{10}$.
a. Proper data table marked with variables and units. (0.3 pts)

| Applied voltage (Volts) | Light intensity (Volts) | Applied voltage <br> (Volts) | Light intensity (Volts) |
| :---: | :---: | :---: | :---: |
| 0.00 | 0.00 | 2.44 | 1.22 |
| 0.10 | 0.00 | 2.50 | 1.26 |
| 0.20 | 0.00 | 2.55 | 1.27 |
| 0.30 | 0.00 | 2.60 | 1.29 |
| 0.40 | 0.00 | 2.67 | 1.32 |
| 0.50 | 0.00 | 2.72 | 1.33 |
| 0.60 | 0.00 | 2.85 | 1.36 |
| 0.70 | 0.00 | 2.97 | 1.37 |
| 0.80 | 0.00 | 3.11 | 1.38 |
| 0.90 | 0.00 | 3.20 | 1.39 |
| 1.00 | 0.00 | 3.32 | 1.39 |
| 1.10 | 0.02 | 3.41 | 1.39 |
| 1.20 | 0.04 | 3.50 | 1.40 |
| 1.24 | 0.04 | 3.60 | 1.39 |
| 1.30 | 0.04 | 3.70 | 1.40 |
| 1.34 | 0.03 | 3.80 | 1.40 |
| 1.38 | 0.02 | 4.03 | 1.40 |
| 1.45 | 0.01 | 4.22 | 1.40 |
| 1.48 | 0.01 | 4.40 | 1.39 |
| 1.55 | 0.02 | 4.61 | 1.39 |
| 1.59 | 0.03 | 4.78 | 1.40 |
| 1.64 | 0.05 | 5.03 | 1.39 |
| 1.71 | 0.11 | 5.20 | 1.39 |
| 1.78 | 0.21 | 5.39 | 1.38 |
| 1.81 | 0.26 | 5.61 | 1.39 |
| 1.85 | 0.33 | 5.81 | 1.38 |
| 1.90 | 0.44 | 6.02 | 1.38 |
| 1.96 | 0.57 | 6.21 | 1.38 |
| 2.03 | 0.70 | 6.40 | 1.38 |
| 2.08 | 0.80 | 6.63 | 1.38 |
| 2.15 | 0.92 | 6.80 | 1.38 |
| 2.21 | 1.02 | 7.02 | 1.38 |
| 2.28 | 1.10 | 7.20 | 1.38 |
| 2.33 | 1.14 |  |  |
| 2.39 | 1.19 |  |  |

b. Properly choose the size of scales and units for abscissa and ordinate that bears the relation to the accuracy and range of the experiment. ( 0.3 pts )
c. Correct measurement of the light intensity ( $\mathcal{J}$ ) as a function of the applied voltage ( $\mathrm{V}_{\mathrm{rms}}$ ) and adequate $\mathcal{J}$ - $\mathrm{V}_{\text {rms }}$ curve plot.

- The intensity of the transmission light is smaller than 0.05 Volts in the normally black mode. (0.4 pts)
- There is a small optical bounce before the external applied voltage reaches the critical voltage. ( 0.8 pts )
- The intensity of the transmission light increases rapidly and abruptly when the external applied voltage exceeds the critical voltage. ( 0.4 pts )
- The intensity of the transmission light displays the plateau behavior as the external applied voltage exceeds 3.0 Volts. ( 0.4 pts)

d. Adequate value of $\gamma$ with error.
- Find the maximum value of the light intensity in the region of the applied voltage between 3.0 and 7.2 Volts ( 0.6 pts)
- Determine the value of $90 \%$ of the maximum light intensity. Obtain the value of the applied voltage $\mathrm{V}_{90}$ by interpolation. ( 0.6 pts )
- Determine the value of $10 \%$ of the maximum light intensity. Obtain the value of the applied voltage $\mathrm{V}_{10}$ by interpolation. ( 0.6 pts )
- Correct $\gamma \pm \Delta \gamma$ value, $(0.42 \sim 0.44) \pm 0.02$. ( $0.4+0.2 \mathrm{pts}$ )


## Question B-(2) (Total 2.5 points)

Determine the critical voltage $\mathrm{V}_{\mathrm{c}}$ of this NB $90^{\circ}$ TN LC cell. Show explicitly with graph how you determine the value $\mathrm{V}_{\mathrm{c}}$.
a. Adequate value of $\mathrm{V}_{\mathrm{C}}$ with error, $\mathrm{V}_{\mathrm{C}} \pm \Delta \mathrm{V}_{\mathrm{C}}$.

- Make the expanded scale plot and take more data points in the region of $\mathrm{V}_{\mathrm{C}}$. (0.8 pts)
- Determine the value of $\mathrm{V}_{\mathrm{C}}$ when the intensity of the transmission light increases rapidly and abruptly. ( 0.7 pts )
- Correct $\mathrm{V}_{\mathrm{C}} \pm \Delta \mathrm{V}_{\mathrm{C}}$ value, (1.20~1.50) $\pm 0.01$ Volts. ( $0.8+0.2 \mathrm{pts}$ )

(The data shown in this graph do not correspond to the data shown on the previous page. This graph only shows how to obtain Vc.)


## Part C: Optical Properties of Nematic Liquid Crystal : Electro-optical switching characteristic of parallel aligned LC cell

Question C-(1) (2.5 points)
Assume that the wavelength of laser light 650 nm , LC layer thickness $7.7 \mu \mathrm{~m}$, and approximate value of $\Delta \mathrm{n} \approx 0.25$ are known. From the experimental data $\mathrm{T}_{\perp}$ and $\mathrm{T}_{\|}$ obtained above, calculate the accurate value of the phase retardation $\delta$ and accurate value of birefringence $\Delta \mathrm{n}$ of this LC cell at $\mathrm{V}=0$.
a. Adequate value of $\delta$ and $\Delta \mathrm{n}$ with error.

- Take and average the values of $\mathrm{T}_{\|}$. ( 0.3 pts )
- Take and average the values of $\mathrm{T}_{\perp}$. ( 0.3 pts )
- Determine the value of order m. ( 0.9 pts )
- Correct $\delta$ value, $15.7 \sim 18.2$. ( 0.5 pts$)$
- Correct $\Delta \mathrm{n}$ value, $0.20 \sim 0.24 \quad(0.5 \mathrm{pts})$
$T_{/ /}=\frac{0.31+0.31+0.31}{3}=0.31 \pm 0.01$ Volts
$T_{\perp}=\frac{1.04+1.03+1.04}{3}=1.04 \pm 0.01$ Volts
$\tan \frac{\delta}{2}= \pm \frac{\sqrt{T_{\perp}}}{\sqrt{T_{/ \prime}}}=-1.83^{*} \quad \therefore \delta=4.14+2 m \pi \quad($ or $-2.14+2 m \pi)$
$\delta=\frac{2 \pi d \Delta n}{\lambda}=\frac{2 \pi \times 7.7 \times 0.25}{0.65}=18.61$
Take $m=2$ (or 3 ) $\quad \therefore \delta=16.70(5.32 \pi)$
From $\delta=\frac{2 \pi d \Delta n}{\lambda} \quad \therefore \Delta n=\frac{\delta \lambda}{2 \pi d}=0.22$
Accepted value for $\therefore \Delta n=(0.20 \sim 0.24)$
*If $\tan \frac{\delta}{2}=1.83$, the value for $\delta$ will be either $4.68 \pi$ or $6.68 \pi$, which is not consistent with data figure of problem C-(2).


## Question C-(2) (Total 3.0 points)

Measure, tabulate, and plot the electro-optical switching curve for $\mathrm{T}_{\|}$of_this parallel aligned LC cell in the $\theta=45^{\circ}$ configuration.
a. Proper data table marked with variables and units. ( 0.3 pts )

| Applied voltage (Volts) | Light intensity (Volts) | Applied voltage (Volts) | Light intensity (Volts) | Applied voltage (Volts) | Light intensity (Volts) |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.00 | 0.30 | 2.01 | 1.47 | 3.33 | 0.00 |
| 0.10 | 0.30 | 2.04 | 1.48 | 3.36 | 0.00 |
| 0.20 | 0.29 | 2.07 | 1.48 | 3.39 | 0.00 |
| 0.30 | 0.29 | 2.10 | 1.48 | 3.42 | 0.00 |
| 0.40 | 0.29 | 2.13 | 1.45 | 3.45 | 0.00 |
| 0.50 | 0.28 | 2.16 | 1.42 | 3.48 | 0.00 |
| 0.60 | 0.26 | 2.19 | 1.38 | 3.51 | 0.00 |
| 0.70 | 0.23 | 2.22 | 1.33 | 3.60 | 0.01 |
| 0.80 | 0.19 | 2.25 | 1.27 | 3.70 | 0.02 |
| 0.90 | 0.09 | 2.28 | 1.20 | 3.80 | 0.03 |
| 0.99 | 0.00 | 2.31 | 1.14 | 3.90 | 0.04 |
| 1.02 | 0.06 | 2.34 | 1.07 | 4.00 | 0.07 |
| 1.05 | 0.16 | 2.37 | 1.00 | 4.10 | 0.09 |
| 1.08 | 0.25 | 2.40 | 0.94 | 4.20 | 0.11 |
| 1.11 | 0.40 | 2.43 | 0.87 | 4.30 | 0.14 |
| 1.14 | 0.67 | 2.46 | 0.79 | 4.40 | 0.16 |
| 1.17 | 0.93 | 2.49 | 0.72 | 4.50 | 0.19 |
| 1.20 | 1.25 | 2.52 | 0.66 | 4.60 | 0.22 |
| 1.26 | 1.31 | 2.55 | 0.61 | 4.70 | 0.25 |
| 1.29 | 1.36 | 2.58 | 0.56 | 4.80 | 0.28 |
| 1.32 | 1.32 | 2.61 | 0.51 | 4.90 | 0.31 |
| 1.35 | 1.09 | 2.64 | 0.46 | 5.01 | 0.34 |
| 1.38 | 0.85 | 2.67 | 0.42 | 5.11 | 0.37 |
| 1.41 | 0.62 | 2.70 | 0.37 | 5.21 | 0.39 |
| 1.44 | 0.46 | 2.73 | 0.33 | 5.29 | 0.42 |
| 1.47 | 0.29 | 2.76 | 0.30 | 5.39 | 0.44 |
| 1.50 | 0.13 | 2.79 | 0.26 | 5.51 | 0.48 |
| 1.53 | 0.06 | 2.82 | 0.23 | 5.57 | 0.49 |
| 1.59 | 0.03 | 2.85 | 0.21 | 5.70 | 0.52 |
| 1.62 | 0.05 | 2.88 | 0.18 | 5.80 | 0.55 |
| 1.65 | 0.15 | 2.91 | 0.16 | 5.90 | 0.57 |
| 1.68 | 0.24 | 2.94 | 0.14 | 6.01 | 0.60 |
| 1.71 | 0.34 | 2.97 | 0.12 | 6.10 | 0.62 |
| 1.74 | 0.49 | 3.00 | 0.09 | 6.19 | 0.64 |
| 1.77 | 0.63 | 3.06 | 0.08 | 6.30 | 0.66 |
| 1.80 | 0.78 | 3.09 | 0.06 | 6.40 | 0.69 |
| 1.83 | 0.92 | 3.12 | 0.05 | 6.60 | 0.73 |
| 1.86 | 1.05 | 3.18 | 0.04 | 6.70 | 0.74 |
| 1.89 | 1.19 | 3.21 | 0.03 | 6.80 | 0.76 |
| 1.92 | 1.27 | 3.24 | 0.02 | 7.00 | 0.80 |
| 1.95 | 1.34 | 3.27 | 0.02 | 7.20 | 0.83 |
| 1.98 | 1.40 | 3.30 | 0.01 |  |  |

b. Properly choose the size of scales and units for abscissa and ordinate that bears the relation to the accuracy and range of the experiment. ( 0.3 pts )
c. Correct measurement of the $\mathrm{T}_{\|}$as a function of the applied voltage $\left(\mathrm{V}_{\mathrm{rms}}\right)$ and adequate $\mathrm{T}_{\|}-\mathrm{V}_{\text {rms }}$ curve plot.

- Three minima and two sharp maxima. (1.5 pts)
- Maxima values within $15 \%$ from each other. ( 0.5 pts )
- Minima are less than the values of 0.1 Volts. ( 0.4 pts)



## Question C-(3) (Total 2.0 points)

From the electro-optical switching data, find the value of the external applied voltage $\mathrm{V}_{\pi}$.
a. Adequate value of $\mathrm{V}_{\pi}$ with error.

- Make the expanded scale plot and take more data points in the region of $\mathrm{V}_{\pi}$. ( 0.3 pts )
- Indicate the correct minimum of $\mathrm{V}_{\pi}$. ( 0.8 pts )
- Obtain the value of $\mathrm{V}_{\pi}$ by interpolation or rounding. ( 0.5 pts )
- Correct $V_{\pi}$ value : $(3.2 \sim 3.5) \pm 0.01$ Volts. $(0.2+0.2$ pts $)$



## Marking Scheme

Part A: Optical Properties of Laser Diode

| No. | Contents | Sub <br> Scores | Total <br> Scores |
| :---: | :--- | :---: | :---: |
| A(1) | Measure, tabulate, and plot the $\mathcal{J}$ vs. $I$ curve. | 1.5 pts. |  |
| a | Proper data table marked with variables and units. | 0.3 |  |
| b | Proper sizes of scales, and units for abscissa and ordinate that bear <br> relation to the accuracy and range of the experiment. | 0.3 |  |
| c | Proper data and adequate curve plotting (Fig. A-1) | 0.9 |  |
| A(2) | Estimate the maximum current $I_{m}$ with uncertainty in the linear region <br> of the $\mathcal{J}$ vs. $I$ curve. Mark the linear region on the $\mathcal{J}-I$ curve figure by <br> using arrows ( $\downarrow$ ) and determine the threshold current $I_{t h}$ with | 3.5 pts. |  |
| uncertainty. |  |  |  |

## Part B: Optical Properties of Nematic Liquid Crystal

Electro-optical switching characteristic of $90^{\circ} \mathrm{TN}$ LC cell

| No. | Contents | Sub <br> Scores | Total <br> Scores |
| :---: | :--- | :---: | :---: |
| B-(1) | Measure, tabulate, and plot the electro-optical switching curve ( $\mathcal{J}$ vs. <br> $\mathrm{V}_{\text {rms }}$ curve) of the NB $90^{\circ} \mathrm{TN} \mathrm{LC} ,\mathrm{and} \mathrm{find} \mathrm{its} \mathrm{switching} \mathrm{slope} \gamma$, <br> where $\gamma$ is defined as $\left(\mathrm{V}_{90}-\mathrm{V}_{10}\right) / \mathrm{V}_{10}$. | 5.0 pts. |  |
| a | Proper data table marked with variables and units. | 0.3 |  |
| b | Properly choose the size of scales and units for abscissa and ordinate <br> that bears the relation to the accuracy and range of the experiment. | 0.3 |  |
| c | Correct measurement of the light intensity (J) as a function of the <br> applied voltage $\left(\mathrm{V}_{\text {rms }}\right)$ and adequate $\mathcal{J}-\mathrm{V}_{\text {rms }}$ curve plot. |  |  |
|  | - The intensity of the transmission light reaches zero value in the <br> normally black mode. | 0.4 |  |
|  | - There is a small optical bounce before the external applied voltage <br> reaches the critical voltage. | 0.8 |  |
|  | - The intensity of the transmission light increases rapidly and | 0.4 |  |


|  | abruptly when the external applied voltage exceeds the critical <br> voltage. |  |  |
| :---: | :--- | :--- | :--- |
|  | - The intensity of the transmission light displays the plateau <br> behavior as the external applied voltage exceeds 3.0 Volts. | 0.4 |  |
| d | Adequate value of $\gamma$ with error, $\gamma \pm \Delta \gamma$. | 0.6 |  |
|  | - Correctly analyzing the maximum light intensity. | 0.6 |  |
|  | - Correctly analyzing the value of $\mathrm{V}_{90}$. | 0.6 |  |
|  | - Correctly analyzing the value of $\mathrm{V}_{10}$. | 0.6 |  |
|  | - Correct $\gamma \pm \Delta \gamma$ value, $(0.42 \sim 0.44) \pm 0.02$. | 2.5 pts. |  |
| B-(2) | Determine the critical voltage $\mathrm{V}_{\mathrm{C}}$ of this $\mathrm{NB} 90^{\circ} \mathrm{TN}$ LC cell. <br> Show explicitly with graph how you determine the value $\mathrm{V}_{\mathrm{c} .}$ |  |  |
|  | Adequate value of $\mathrm{V}_{\mathrm{C}}$ with error, $\mathrm{V}_{\mathrm{C}} \pm \Delta \mathrm{V}_{\mathrm{C}}$. 0.8 <br>  - Make the expanded scale plot and take more data points in the <br> region of $\mathrm{V}_{\mathrm{C}}$.  |  |  |
|  | - Correctly analyzing the value of $\mathrm{V}_{\mathrm{C}}$. | 0.7 |  |
|  | - Correct $\mathrm{V}_{\mathrm{C}} \pm \Delta \mathrm{V}_{\mathrm{C}}$ value, $(1.2 \sim 1.5) \pm 0.01$ Volts. | 1.0 |  |

Part C: Optical Properties of Nematic Liquid Crystal :
Electro-optical switching characteristic of parallel aligned LC cell

| No. | Contents | Sub <br> Scores | Total <br> Scores |
| :--- | :--- | :---: | :---: |
| C-(1) | Assume that the wavelength of laser light 650 nm, LC layer thickness <br> $7.7 \mu \mathrm{~m}$, and approximate value of $\Delta \mathrm{n} \approx 0.25$ are known. From the <br> experimental data $\mathrm{T}_{\perp}$ and $\mathrm{T}_{\\|}$obtained above, calculate the accurate <br> value of the phase retardation $\delta$ and accurate value of birefringence <br> $\Delta \mathrm{n}$ of this LC cell at $\mathrm{V}=0$. | 2.5 pts. |  |
|  | Adequate value of $\delta$ and $\Delta \mathrm{n}$ with error. |  |  |
|  | - Correctly analyzing the values of $\mathrm{T}_{\\|}$. | 0.3 |  |
|  | - Correctly analyzing the values of $\mathrm{T}_{\perp}$. | 0.3 |  |
|  | - Correctly determining the value of order m. | 0.9 |  |
|  | - Correct $\delta$ value, $17.7 \sim 18.2$. | 0.5 | 0.5 |
|  | - Correct $\Delta \mathrm{n}$ value, $0.23 \sim 0.25$. | 3.0 pts. |  |
| C-(2) | Measure, tabulate, and plot the electro-optical switching curve for T <br> of_this parallel aligned LC cell in the $\theta=45^{\circ}$ configuration. |  |  |
| a | Proper data table marked with variables and units. | 0.3 |  |
| b | Properly choose the size of scales and units for abscissa and ordinate | 0.3 |  |


|  | that bears the relation to the accuracy and range of the experiment. |  |  |
| :---: | :---: | :---: | :---: |
| c | Correct measurement of the $\mathrm{T}_{\\|}$as a function of the applied voltage $\left(\mathrm{V}_{\text {rms }}\right)$ and adequate $\mathrm{T}_{\\|}-\mathrm{V}_{\text {rms }}$ curve plot. |  |  |
|  | - Three minima and two sharp maxima. | 1.5 |  |
|  | - Maxima values within 15 \% from each other. | 0.5 |  |
|  | - Minima are less than the values of 0.1 Volts. | 0.4 |  |
| C-(3) | From the electro-optical switching data, find the value of the external applied voltage $\mathrm{V}_{\pi}$ |  | 2.0 pts. |
|  | Adequate value of $\mathrm{V}_{\pi}$ with error. |  |  |
|  | - Make the expanded scale plot and take more data points in the region of $\mathrm{V}_{\pi}$. | 0.3 |  |
|  | - Indicate the correct minimum of $\mathrm{V}_{\pi}$. | 0.8 |  |
|  | - Correctly analyzing the value of $\mathrm{V}_{\pi}$. | 0.5 |  |
|  | - Correct $\mathrm{V}_{\pi} \pm \Delta \mathrm{V}_{\pi}$ value, ( $3.2 \sim 3.5$ ) $\pm 0.1$ Volts. | 0.4 |  |

## Theoretical Question 1:

## "Ping-Pong" Resistor

A capacitor consists of two circular parallel plates both with radius $R$ separated by distance $d$, where $d \ll R$, as shown in Fig. 1.1(a). The top plate is connected to a constant voltage source at a potential $V$ while the bottom plate is grounded. Then a thin and small disk of mass $m$ with radius $r(\ll R, d)$ and thickness $t(\ll r)$ is placed on the center of the bottom plate, as shown in Fig. 1.1(b).

Let us assume that the space between the plates is in vacuum with the dielectric constant $\varepsilon_{0}$; the plates and the disk are made of perfect conductors; and all the electrostatic edge effects may be neglected. The inductance of the whole circuit and the relativistic effects can be safely disregarded. The image charge effect can also be neglected.


Figure 1.1 Schematic drawings of (a) a parallel plate capacitor connected to a constant voltage source and (b) a side view of the parallel plates with a small disk inserted inside the capacitor. (See text for details.)
(a) [1.2 points] Calculate the electrostatic force $F_{\mathrm{p}}$ between the plates separated by $d$ before inserting the disk in-between as shown in Fig. 1.1(a).
(b) [0.8 points] When the disk is placed on the bottom plate, a charge $q$ on the disk of Fig. 1.1(b) is related to the voltage $V$ by $q=\chi V$. Find $\chi$ in terms of $r, d$, and $\varepsilon_{0}$.
(c) [0.5 points] The parallel plates lie perpendicular to a uniform gravitational field $g$. To lift up the disk at rest initially, we need to increase the applied voltage beyond a
threshold voltage $V_{\text {th }}$. Obtain $V_{\text {th }}$ in terms of $m, g, d$, and $\chi$.
(d) [2.3 points] When $V>V_{\text {th }}$, the disk makes an up-and-down motion between the plates. (Assume that the disk moves only vertically without any wobbling.) The collisions between the disk and the plates are inelastic with the restitution coefficient $\eta \equiv\left(\mathrm{v}_{\text {after }} / \mathrm{v}_{\text {before }}\right)$, where $\mathrm{v}_{\text {before }}$ and $\mathrm{v}_{\text {after }}$ are the speeds of the disk just before and after the collision respectively. The plates are stationarily fixed in position. The speed of the disk just after the collision at the bottom plate approaches a "steady-state speed" $\mathrm{v}_{\mathrm{s}}$, which depends on $V$ as follows:

$$
\begin{equation*}
\mathrm{v}_{\mathrm{s}}=\sqrt{\alpha V^{2}+\beta} . \tag{1.1}
\end{equation*}
$$

Obtain the coefficients $\alpha$ and $\beta$ in terms of $m, g, \chi, d$, and $\eta$. Assume that the whole surface of the disk touches the plate evenly and simultaneously so that the complete charge exchange happens instantaneously at every collision.
(e) [2.2 points] After reaching its steady state, the time-averaged current $I$ through the capacitor plates can be approximated by $I=\gamma V^{2}$ when $q V \gg m g d$. Express the coefficient $\gamma$ in terms of $m, \chi, d$, and $\eta$.
(f) [3 points] When the applied voltage $V$ is decreased (extremely slowly), there exists a critical voltage $V_{c}$ below which the charge will cease to flow. Find $V_{c}$ and the corresponding current $I_{\mathrm{c}}$ in terms of $m, g, \chi, d$, and $\eta$. By comparing $V_{c}$ with the lift-up threshold $V_{\text {th }}$ discussed in (c), make a rough sketch of the $I-V$ characteristics when $V$ is increased and decreased in the range from $V=0$ to $3 V_{\text {th }}$.

| Country <br> Code | Student <br> Code | Question <br> Number |
| :---: | :---: | :---: |
|  |  | 1 |

## Answer Form

## Theoretical Question 1:

(a) $F_{\mathrm{p}}=$ $\square$
(b) $\chi=$

(c) $V_{\mathrm{th}}=$

(d)

(e) $\gamma=$

(f) $I_{c}=$


## Theoretical Question 2

## Rising Balloon

A rubber balloon filled with helium gas goes up high into the sky where the pressure and temperature decrease with height. In the following questions, assume that the shape of the balloon remains spherical regardless of the payload, and neglect the payload volume. Also assume that the temperature of the helium gas inside of the balloon is always the same as that of the ambient air, and treat all gases as ideal gases. The universal gas constant is $R=8.31 \mathrm{~J} / \mathrm{mol} \cdot \mathrm{K}$ and the molar masses of helium and air are $M_{H}=4.00 \times 10^{-3} \mathrm{~kg} / \mathrm{mol}$ and $M_{A}=28.9 \times 10^{-3} \mathrm{~kg} / \mathrm{mol}$, respectively. The gravitational acceleration is $g=9.8 \mathrm{~m} / \mathrm{s}^{2}$.

## [Part A ]

(a) [1.5 points] Let the pressure of the ambient air be $P$ and the temperature be $T$. The pressure inside of the balloon is higher than that of outside due to the surface tension of the balloon. The balloon contains $n$ moles of helium gas and the pressure inside is $P+\Delta P$. Find the buoyant force $F_{\mathrm{B}}$ acting on the balloon as a function of $P$ and $\Delta P$.
(b) [2 points] On a particular summer day in Korea, the air temperature $T$ at the height $z$ from the sea level was found to be $T(z)=T_{0}\left(1-z / z_{0}\right)$ in the range of $0<z<15$ km with $z_{0}=49 \mathrm{~km}$ and $T_{0}=303 \mathrm{~K}$. The pressure and density at the sea level were $P_{0}$ $=1.0 \mathrm{~atm}=1.01 \times 10^{5} \mathrm{~Pa}$ and $\rho_{0}=1.16 \mathrm{~kg} / \mathrm{m}^{3}$, respectively. For this height range, the pressure takes the form

$$
\begin{equation*}
P(z)=P_{0}\left(1-z / z_{0}\right)^{\eta} . \tag{2.1}
\end{equation*}
$$

Express $\eta$ in terms of $z_{0}, \rho_{0}, P_{0}$, and $g$, and find its numerical value to the two significant digits. Treat the gravitational acceleration as a constant, independent of height.

## [Part B ]

When a rubber balloon of spherical shape with un-stretched radius $r_{0}$ is inflated to a sphere of radius $r\left(\geq r_{0}\right)$, the balloon surface contains extra elastic energy due to the stretching. In a simplistic theory, the elastic energy at constant temperature $T$ can be expressed by

$$
\begin{equation*}
U=4 \pi r_{0}^{2} \kappa R T\left(2 \lambda^{2}+\frac{1}{\lambda^{4}}-3\right) \tag{2.2}
\end{equation*}
$$

where $\lambda \equiv r / r_{0}(\geq 1)$ is the size-inflation ratio and $\kappa$ is a constant in units of $\mathrm{mol} / \mathrm{m}^{2}$.
(c) [2 points] Express $\Delta P$ in terms of parameters given in Eq. (2.2), and sketch $\Delta P$ as a function of $\lambda=r / r_{0}$.
(d) [1.5 points] The constant $\kappa$ can be determined from the amount of the gas needed to inflate the balloon. At $T_{0}=303 \mathrm{~K}$ and $P_{0}=1.0 \mathrm{~atm}=1.01 \times 10^{5} \mathrm{~Pa}$, an un-stretched balloon ( $\lambda=1$ ) contains $n_{0}=12.5$ moles of helium. It takes $n=3.6 n_{0}=45$ moles in total to inflate the balloon to $\lambda=1.5$ at the same $T_{0}$ and $P_{0}$. Express the balloon parameter $a$, defined as $a=\kappa / \kappa_{0}$, in terms of $n, n_{0}$, and $\lambda$, where $\kappa_{0} \equiv \frac{r_{0} P_{0}}{4 R T_{0}}$. Evaluate $a$ to the two significant digits.

## [Part C]

A balloon is prepared as in (d) at the sea level (inflated to $\lambda=1.5$ with $n=3.6 n_{0}=45$ moles of helium gas at $T_{0}=303 \mathrm{~K}$ and $P_{0}=1 \mathrm{~atm}=1.01 \times 10^{5} \mathrm{~Pa}$ ). The total mass including gas, balloon itself, and other payloads is $M_{\mathrm{T}}=1.12 \mathrm{~kg}$. Now let the balloon rise from the sea level.
(e) [3 points] Suppose that the balloon eventually stops at the height $z_{f}$ where the buoyant force balances the total weight. Find $z_{f}$ and the inflation ratio $\lambda_{f}$ at that
height. Give the answers in two significant digits. Assume there are no drift effect and no gas leakage during the upward flight.

IPhO
A Theoretical Question 2 / Answer Form

| Country <br> Code | Student <br> Code | Question <br> Number |
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## Answer Form

Theoretical Question 2:
(a) $F_{B}=$

(b) $\eta=$


Numerical value of $\eta=$

(c) $\Delta P=$


(d) $a=$


Numerical value of $a=$

(e) $z_{f}=$


## Theoretical Question 3

## Atomic Probe Microscope

Atomic probe microscopes (APMs) are powerful tools in the field of nano-science. The motion of a cantilever in APM can be detected by a photo-detector monitoring the reflected laser beam, as shown in Fig. 3.1. The cantilever can move only in the vertical direction and its displacement $z$ as a function of time $t$ can be described by the equation

$$
\begin{equation*}
m \frac{d^{2} z}{d t^{2}}+b \frac{d z}{d t}+k z=F \tag{3.1}
\end{equation*}
$$

where $m$ is the cantilever mass, $k=m \omega_{0}^{2}$ is the spring constant of the cantilever, $b$ is a small damping coefficient satisfying $\omega_{0} \gg(b / m)>0$, and finally $F$ is an external driving force of the piezoelectric tube.


Figure 3.1 A schematic diagram for a scanning probe microscope (SPM). The inset in the lower right corner represents a simplified mechanical model to describe the coupling of the piezotube with the cantilever.

## [Part A]

(a) [1.5 points] When $F=F_{0} \sin \omega t, z(t)$ satisfying Eq. (3.1) can be written as $z(t)=A \sin (\omega t-\phi)$, where $A>0$ and $0 \leq \phi \leq \pi$. Find the expression of the
amplitude $A$ and $\tan \phi$ in terms of $F_{0}, m, \omega, \omega_{0}$, and $b$. Obtain $A$ and the phase $\phi$ at the resonance frequency $\omega=\omega_{0}$.
(b) [1 point] A lock-in amplifier shown in Fig. 3.1 multiplies an input signal by the lockin reference signal, $V_{R}=V_{R 0} \sin \omega t$, and then passes only the dc (direct current) component of the multiplied signal. Assume that the input signal is given by $V_{i}=V_{i 0} \sin \left(\omega_{i} t-\phi_{i}\right)$. Here $V_{R 0}, V_{i 0}, \omega_{i}$, and $\phi_{i}$ are all positive given constants. Find the condition on $\omega(>0)$ for a non-vanishing output signal. What is the expression for the magnitude of the non-vanishing dc output signal at this frequency?
(c) [1.5 points] Passing through the phase shifter, the lock-in reference voltage $V_{R}=V_{R 0} \sin \omega t$ changes to $V_{R}^{\prime}=V_{R 0} \sin (\omega t+\pi / 2) . V_{R}^{\prime}$, applied to the piezoelectric tube, drives the cantilever with a force $F=c_{1} V_{R}^{\prime}$. Then, the photo-detector converts the displacement of the cantilever, $z$, into a voltage $V_{i}=c_{2} z$. Here $c_{1}$ and $c_{2}$ are constants. Find the expression for the magnitude of the dc output signal at $\omega=\omega_{0}$.
(d) [2 points] The small change $\Delta m$ of the cantilever mass shifts the resonance frequency by $\Delta \omega_{0}$. As a result, the phase $\phi$ at the original resonance frequency $\omega_{0}$ shifts by $\Delta \phi$. Find the mass change $\Delta m$ corresponding to the phase shift $\Delta \phi=\pi / 1800$, which is a typical resolution in phase measurements. The physical parameters of the cantilever are given by $m=1.0 \times 10^{-12} \mathrm{~kg}, k=1.0 \mathrm{~N} / \mathrm{m}$, and $(b / m)=1.0 \times 10^{3} \quad \mathrm{~s}^{-1}$. Use the approximations $(1+x)^{a} \approx 1+a x$ and $\tan (\pi / 2+x) \approx-1 / x$ when $|x| \ll 1$.

## [Part B]

From now on let us consider the situation that some forces, besides the driving force discussed in Part A, act on the cantilever due to the sample as shown in Fig.3.1.
(e) [1.5 points] Assuming that the additional force $f(h)$ depends only on the distance $h$ between the cantilever and the sample surface, one can find a new equilibrium position $h_{0}$. Near $h=h_{0}$, we can write $f(h) \approx f\left(h_{0}\right)+c_{3}\left(h-h_{0}\right)$, where $c_{3}$ is a constant in $h$. Find the new resonance frequency $\omega_{0}^{\prime}$ in terms of $\omega_{0}, m$, and $c_{3}$.
(f) [2.5 points] While scanning the surface by moving the sample horizontally, the tip of the cantilever charged with $Q=6 e$ encounters an electron of charge $q=e$ trapped
(localized in space) at some distance below the surface. During the scanning around the electron, the maximum shift of the resonance frequency $\Delta \omega_{0}\left(=\omega_{0}^{\prime}-\omega_{0}\right)$ is observed to be much smaller than $\omega_{0}$. Express the distance $d_{0}$ from the cantilever to the trapped electron at the maximum shift in terms of $m, q, Q, \omega_{0}, \Delta \omega_{0}$, and the Coulomb constant $k_{e}$. Evaluate $d_{0}$ in $\mathrm{nm}\left(1 \mathrm{~nm}=1 \times 10^{-9} \mathrm{~m}\right)$ for $\Delta \omega_{0}=20 \mathrm{~s}^{-1}$.
The physical parameters of the cantilever are $m=1.0 \times 10^{-12} \mathrm{~kg}$ and $k=1.0 \mathrm{~N} / \mathrm{m}$. Disregard any polarization effect in both the cantilever tip and the surface. Note that $k_{e}=1 / 4 \pi \varepsilon_{0}=9.0 \times 10^{9} \mathrm{~N} \cdot \mathrm{~m}^{2} / \mathrm{C}^{2}$ and $e=-1.6 \times 10^{-19} \mathrm{C}$.

| Country <br> Code | Student <br> Code | Question <br> Number |
| :---: | :---: | :---: |
|  |  | 3 |

## Answer Form

## Theoretical Question 3:

(a) $A=$
 and

At $\omega=\omega_{0}, \quad A=$

and $\phi=$

(b) The condition on $\omega$ for a non-vanishing output signal :


The magnitude of the dc signal $=\square$
(c) The magnitude of the signal $=$

(d) $\Delta m=$


| Country <br> Code | Student <br> Code | Question <br> Number |
| :---: | :---: | :---: |
|  |  | 3 |

(e) $\omega_{0}^{\prime}=$

(f) $d_{0}=$



Pohang, Korea
15 ~ 23 July 2004

## Experimental Competition <br> Monday, 19 July 2004

## Please, first read the following instruction carefully:

1. The time available is 5 hours.
2. Use only the pen provided.
3. Use only the front side of the writing sheets. Write only inside the boxed area.
4. In addition to the blank writing sheets, there are Answer Forms where you must summarize the results you have obtained.
5. Write on the blank writing sheets the results of your measurements and whatever else you consider is required for the solution to the question. Please, use as little text as possible; express yourself primarily in equations, numbers, figures, and plots.
6. In the boxes at the top of each sheet of paper write down your country code (Country Code) and student number (Student Code). In addition, on each blank writing sheets, write down the progressive number of each sheet (Page Number) and the total number of writing sheets used (Total Number of Pages). If you use some blank writing sheets for notes that you do not wish to be marked, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the experiment, arrange all sheets in the following order:

- Answer forms (top)
- used writing sheets in order
- the sheets you do not wish to be marked
- unused writing sheets
- the printed question (bottom)

8. It is not necessary to specify the error range of your values. However, their deviations from the actual values will determine your mark.
9. Place the papers inside the envelope and leave everything on your desk. You are not allowed to take anv sheet of paper or anv material used in the experiment out of the room.

## Apparatus and materials

1. List of available apparatus and materials

|  | Name | Quantity |  | Name | Quantity |
| :---: | :---: | :---: | :---: | :---: | :---: |
| A | Photogate timer | 1 | L | Philips screw driver | 1 |
| B | Photogate | 1 | M | Weight with a string | 1 |
| C | Connecting cable | 1 | N | Electronic balance | 1 |
| D | Mechanical "black box" (Black cylinder) | 1 | O | Stand with a ruler | 1 |
| E | Rotation stage | 1 | P | U-shaped support | 1 |
| F | Rubber pad | 1 | Q | C-clamp | 1 |
| G | Pulley | 2 |  | Ruler ( $0.50 \mathrm{~m}, 0.15 \mathrm{~m}$ ) | 1 each |
| H | Pin | 2 |  | Vernier calipers | 1 |
| I | U-shaped plate | 1 |  | Scissors | 1 |
| J | Screw | 2 |  | Thread | 1 |
| K | Allen (hexagonal, Lshaped) wrench | 1 |  | Spares (string, thread, pin, screw, Allen wrench) |  |


2. Instruction for the Photogate Timer

The Photogate consists of an infrared LED and a photodetector. By connecting the Photogate to the Photogate Timer, you can measure the time duration related to the blocking of the infrared light reaching the sensor.

- Be sure that the Photogate is connected to the Photogate Timer. Turn on the power by pushing the button labelled "POWER".
- To measure the time duration of a single blocking event, push the button labelled "GATE". Use this "GATE" mode for speed measurements.
- To measure the time interval between two or three successive blocking events, push the corresponding "PERIOD". Use this "PERIOD" mode for oscillation measurements.
- If "DELAY" button is pushed in, the Photogate Timer displays the result of each measurement for 5 seconds and then resets itself.
- If "DELAY" button is pushed out, the Photogate Timer displays the result of the previous measurement until the next measurement is completed.
- After any change of button position, press the "RESET" button once to activate the mode change.

Caution: Do not look directly into the Photogate. The invisible infrared light may be harmful to your eyes.


Photogate, Photogate Timer, and connection cable
3. Instruction for the Electronic Balance

- Adjust the bottom legs to set the balance stable. (Although there is a level indicator, setting the balance in a completely horizontal position is not necessary.)
- Without putting anything on the balance, turn it on by pressing the "On/Off" button.
- Place an object on the round weighing pan. Its mass will be displayed in grams.
- If there is nothing on the weighing pan, the balance will be turned off automatically in about 25 seconds.


Balance
4. Instruction for the Rotation Stage

- Adjust the bottom legs to set the rotation stage stable on a rubber pad in a near horizontal position.
- With a U-shaped plate and two screws, mount the Mechanical "Black Box" (black cylinder) on the top of the rotating stub. Use Allen (hexagonal, L-shaped) wrench to tighten the screws.
- The string attached to the weight is to be fixed to the screw on the side of the rotating stub. Use the Philips screw driver.

Caution: Do not look too closely at the Mechanical "Black Box" while it is rotating. Your eyes may get hurt.


Mechanical "Black Box" and rotation stage


Rotating stub


Weight with a string

## Mechanical "Black Box"

## [Question] Find the mass of the ball and the spring constants of two springs in the Mechanical "Black Box".

## General Information on the Mechanical "Black Box"

The Mechanical "Black Box" (MBB) consists of a solid ball attached to two springs in a black cylindrical tube as shown in Fig. 1. The two springs are fashioned from the same tightly wound spring with different number of turns. The masses and the lengths of the springs when they are not extended can be ignored. The tube is homogeneous and sealed with two identical end caps. The part of the end caps plugged into the tube is 5 mm long. The radius of the ball is 11 mm and the inner diameter of the tube is 23 mm . The gravitational acceleration is given as $g=9.8 \mathrm{~m} / \mathrm{s}^{2}$. There is a finite friction between the ball and the inner walls of the tube.


Fig. 1 Mechanical "Black Box" (not to scale)

The purpose of this experiment is to find out the mass $m$ of the ball and the spring constants $k_{1}$ and $k_{2}$ of the springs without opening the MBB. The difficult aspect of this problem is that any single experiment cannot provide the mass $m$ or the position $l$ of the ball because the two quantities are interconnected. Here, $l$ is the distance between the centers of the tube and the ball when the MBB lies horizontally in equilibrium when the friction is zero.

The symbols listed below should be used to represent the physical quantities of interest. If you need to use other physical quantities, use symbols different from those already assigned below to avoid confusion.

## Assigned Physical Symbols

Mass of the ball: $m$
Radius of the ball: $r(=11 \mathrm{~mm})$
Mass of the MBB excluding the ball: $M$
Length of the black tube: $L$
Length of each end cap extending into the tube: $\delta(=5.0 \mathrm{~mm})$
Distance from the center-of-mass of the MBB to the center of the tube: $l_{\mathrm{CM}}$
Distance between the center of the ball and the center of the tube: $x$ (or $l$ at equilibrium when the MBB is horizontal)
Gravitational acceleration: $g\left(=9.8 \mathrm{~m} / \mathrm{s}^{2}\right)$
Mass of the weight attached to a string: $m_{\mathrm{o}}$
Speed of the weight: $v$
Downward displacement of the weight: $h$
Radius of the rotating stub where the string is to be wound: $R$
Moments of inertia: $I, I_{0}, I_{1}, I_{2}$, and so on
Angular velocity and angular frequencies: $\omega, \omega_{1}, \omega_{2}$, and so on
Periods of oscillation: $T_{1}, T_{2}$
Effective total spring constant: $k$
Spring constants of the two springs: $k_{1}, k_{2}$
Number of turns of the springs: $N_{1}, N_{2}$

Caution: Do not try to open the MBB. If you open it, you will be disqualified and your mark in the Experimental Competition will be zero.

Caution: Do not shake violently nor drop the MBB. The ball may be detached from the springs. If your MBB seems faulty, report to the proctors immediately. It will be replaced only once without affecting your mark. Any further replacement will cut down your mark by 0.5 points each time.

## PART-A Product of the mass and the position of the ball $(\boldsymbol{m} \times \boldsymbol{l})$ (4.0 points)

$l$ is the position of the center of the ball relative to that of the tube when the MBB lies horizontally in equilibrium as in Fig. 1. Find the value of the product of the mass $m$ and the position $l$ of the ball experimentally. You will need this to determine the value of $m$ in PART-B.

1. Suggest and justify, by using equations, a method allowing to obtain $m \times l$. (2.0 points)
2. Experimentally determine the value of $m \times l$. (2.0 points)

## PART-B The mass $\boldsymbol{m}$ of the ball (10.0 points)

Figure 2 shows the MBB fixed horizontally on the rotating stub and a weight attached to one end of a string whose other end is wound on the rotating stub. When the weight falls, the string unwinds, and the MBB rotates. By combining the equation pertinent to this experiment with the one obtained in PART-A, you can find an equation for $m$.

Between the ball and the inner walls of the cylindrical tube acts a frictional force. The physical mechanisms of the friction and the slipping of the ball under the rotational motion are complicated. To simplify the analysis, you may ignore the energy dissipation due to kinetic friction.


Fig. 2 Rotation of the Mechanical "Black Box" (not to scale) The angular velocity $\omega$ of the MBB can be obtained from the speed $v$ of the weight passing through the Photogate. $x$ is the position of the ball relative to the rotation axis, and $d$ is the length of the weight.

1. Measure the speed of the weight $v$ for various values of downward displacement $h$ of the weight. It is recommended to scan the whole range from $h=1.0 \times 10^{-2} \mathrm{~m}$ to $4.0 \times 10^{-1} \mathrm{~m}$ by measuring $v$ just once at each $h$ with an interval of $1.0 \times 10^{-2}$ $\sim 2.0 \times 10^{-2} \mathrm{~m}$. Plot the data on graph paper in a form that is suitable to find the value of $m$. After you get a general idea of the relation between $v$ and $h$, you may repeat the measurement or add some data points, if necessary. When the MBB rotates slowly, the ball does not slip from its static equilibrium position because of the friction between the ball and the tube. When the MBB rotates sufficiently fast, the ball hits and actually stays at the end cap of the tube because the springs are weak. Identify the slow rotation region and the fast rotation region on the graph. (4.0 points)
2. Show your measurements are consistent with the fact that $h$ is proportional to $v^{2}$ ( $h=C v^{2}$ ) in the slow rotation region. Show from your measurements that $h=$ $A v^{2}+B$ in the fast rotation region. (1.0 points)
3. The moment of inertia of a ball of radius $r$ and mass $m$ about the axis passing through its center is $2 \mathrm{mr}^{2} / 5$. If the ball is displaced a distance $a$ perpendicular to the axis, the moment of inertia increases by $m a^{2}$. Use the symbol $I$ to represent the total moment of inertia of all the rotating bodies excluding the ball. Relate the coefficient $C$ to the parameters of the MBB such as $m, l$, etc. ( 1.0 points)
4. Relate the coefficients $A$ and $B$ to the parameters of the MBB such as $m, l$, etc. (1.0 points)
5. Determine the value of $m$ from your measurements and the results obtained in PART-A. (3.0 points)

PART-C The spring constants $\boldsymbol{k}_{1}$ and $\boldsymbol{k}_{\mathbf{2}}$ (6.0 points)

In this part, you need to perform small oscillation experiments using the MBB as a rigid pendulum. There are two small holes at each end of the MBB. Two thin pins inserted into the holes can be used as the pivot of small oscillation. The U-shaped support is to be clamped to the stand and used to support the pivot. Note that the angular frequency $\omega$ of small oscillation is given as $\omega=$ [torque/(moment of inertia $\times$ angle) $]^{1 / 2}$. Here, the torque and the moment of inertia are with respect to the pivot. Similarly to PART-B, consider two experimental conditions, shown in Fig. 3, to avoid the unknown moment of inertia $I_{0}$ of the MBB excluding the ball.

(1)

(2)

Fig. 3 Oscillation of the Mechanical "Black Box" (not to scale) The periods of small oscillation, $T_{1}$ and $T_{2}$, for two configurations shown above can be measured using the Photogate. Two pins and a U-shaped support are supplied for this experiment.

1. Measure the periods $T_{1}$ and $T_{2}$ of small oscillation shown in Figs 3(1) and (2) and write down their values, respectively. (1.0 points)
2. Explain (by using equations) why the angular frequencies $\omega_{1}$ and $\omega_{2}$ of small oscillation of the configurations are different. Use the symbol $I_{0}$ to represent the moment of inertia of the MBB excluding the ball for the axis perpendicular to the MBB at the end. Use the symbol $\Delta l$ as the displacement of the ball from the horizontal equilibrium position. (1.0 points)
3. Evaluate $\Delta l$ by eliminating $I_{\mathrm{o}}$ from the previous results. (1.0 points)
4. By combining the results of PART-C 1~3 and PART-B, find and write down the value of the effective total spring constant $k$ of the two-spring system. ( 2.0 points)
5. Obtain the respective values of $k_{1}$ and $k_{2}$. Write down their values. (1.0 points)

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## Answer Form

## PART-A

1. Suggest and justify, by using equations, a method allowing to obtain $m \times l$. (2.0 points)
2. Experimentally determine the value of $m \times l$. ( 2.0 points)

$$
m \times l=
$$

| Country Code | Student Code |
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## PART-B

1. Measure $v$ for various values of $h$. Plot the data on a graph paper in a form that is suitable to find the value of $m$. Identify the slow rotation region and the fast rotation region on the graph. (4.0 points)
(On a separate graph paper)
2. Show from your measurements that $h=C v^{2}$ in the slow rotation region, and $h=$ $A v^{2}+B$ in the fast rotation region. (1.0 points) (In the plot above)
3. Relate the coefficient $C$ to the parameters of the MBB. (1.0 points)

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4. Relate the coefficients $A$ and $B$ to the parameters of the MBB. (1.0 points)

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| :---: | :---: |
|  |  |

5. Determine the value of $m$ from your measurements and the results obtained in PART-A. (3.0 points)

| Country Code | Student Code |
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## PART-C

1. Measure the periods $T_{1}$ and $T_{2}$ of small oscillation shown in Figs. 3 (1) and (2) and write down their values, respectively. (1.0 points)

$$
T_{1}=
$$

$\qquad$ .
$T_{2}=$ $\qquad$ .
2. Explain, by using equations, why the angular frequencies $\omega_{1}$ and $\omega_{2}$ of small oscillation of the configurations are different. (1.0 points)

| Country Code | Student Code |
| :---: | :---: |
|  |  |

3. Evaluate $\Delta l$ by eliminating $I_{\mathrm{o}}$ from the previous results. (1.0 points)

| Country Code | Student Code |
| :---: | :---: |
|  |  |

4. Write down the value of the effective total spring constant $k$ of the two-spring system. (2.0 points)

$$
k=
$$

5. Obtain the respective values of $k_{1}$ and $k_{2}$. Write down their values. (1.0 points)

$$
k_{1}=
$$

$$
k_{2}=
$$

## Theoretical Question 1: Ping-Pong Resistor

## 1. Answers

(a) $\quad F_{\mathrm{R}}=-\frac{1}{2} \pi R^{2} \varepsilon_{0} \frac{V^{2}}{d^{2}}$
(b) $\chi=-\varepsilon_{0} \frac{\pi r^{2}}{d}$
(c) $V_{\text {th }}=\sqrt{\frac{2 m g d}{\chi}}$
(d) $\mathrm{v}_{\mathrm{s}}=\sqrt{\alpha V^{2}+\beta}$

$$
\alpha=\left(\frac{\eta^{2}}{1-\eta^{2}}\right)\left(\frac{2 \chi}{m}\right), \quad \beta=\left(\frac{\eta^{2}}{1+\eta^{2}}\right)(2 g d)
$$

(e) $\gamma=\sqrt{\frac{1+\eta}{1-\eta}} \sqrt{\frac{\chi^{3}}{2 m d^{2}}}$
(f) $\quad V_{c}=\sqrt{\frac{1-\eta^{2}}{1+\eta^{2}}} \sqrt{\frac{m g d}{\chi}}, \quad I_{c}=\frac{2 \eta \sqrt{1-\eta^{2}}}{(1+\eta)\left(1+\eta^{2}\right)} g \sqrt{m \chi}$


## 2. Solutions

(a) [1.2 points]

The charge $Q$ induced by the external bias voltage $V$ can be obtained by applying the Gauss law:

$$
\begin{gather*}
\varepsilon_{0} \oint \vec{E} \cdot d \vec{s}=Q  \tag{a1}\\
Q=\varepsilon_{0} E \cdot\left(\pi R^{2}\right)=\varepsilon_{0}\left(\frac{V}{d}\right) \cdot\left(\pi R^{2}\right), \tag{a2}
\end{gather*}
$$

where $\quad V=E d$.
The energy stored in the capacitor:

$$
\begin{equation*}
U=\int_{0}^{V} Q\left(V^{\prime}\right) d V^{\prime}=\int_{0}^{V} \varepsilon_{0} \pi R^{2}\left(\frac{V^{\prime}}{d}\right) d V^{\prime}=\frac{1}{2} \varepsilon_{0} \pi R^{2} \frac{V^{2}}{d} . \tag{a3}
\end{equation*}
$$

The force acting on the plate, when the bias voltage $V$ is kept constant:

$$
\begin{equation*}
\therefore F_{\mathrm{R}}=+\frac{\partial U}{\partial d}=-\frac{1}{2} \varepsilon_{0} \pi R^{2} \frac{V^{2}}{d^{2}} \text {. } \tag{a4}
\end{equation*}
$$

## [An alternative solution:]

Since the electric field $E^{\prime}$ acting on one plate should be generated by the other plate and its magnitude is

$$
\begin{equation*}
E^{\prime}=\frac{1}{2} E=\frac{V}{2 d}, \tag{a5}
\end{equation*}
$$

the force acting on the plate can be obtained by

$$
\begin{equation*}
F_{\mathrm{R}}=Q E^{\prime} . \tag{a6}
\end{equation*}
$$

(b) [0.8 points]

The charge $q$ on the small disk can also be calculated by applying the Gauss law:

$$
\begin{equation*}
\varepsilon_{0} \oint \vec{E} \cdot d \vec{s}=q . \tag{b1}
\end{equation*}
$$

Since one side of the small disk is in contact with the plate,

$$
\begin{equation*}
q=-\varepsilon_{0} E \cdot\left(\pi r^{2}\right)=-\varepsilon_{0} \frac{\pi r^{2}}{d} V=\chi V \tag{b2}
\end{equation*}
$$

Alternatively, one may use the area ratio for $q=-\left(\frac{\pi r^{2}}{\pi R^{2}}\right) Q$.

$$
\begin{equation*}
\therefore \chi=-\varepsilon_{0} \frac{\pi r^{2}}{d} . \tag{b3}
\end{equation*}
$$

(c) $[0.5$ points $]$

The net force, $F_{\text {net }}$, acting on the small disk should be a sum of the gravitational and electrostatic forces:

$$
\begin{equation*}
F_{\mathrm{net}}=F_{\mathrm{g}}+F_{\mathrm{e}} . \tag{c1}
\end{equation*}
$$

The gravitational force: $F_{\mathrm{g}}=-m g$.
The electrostatic force can be derived from the result of (a) above:

$$
\begin{equation*}
F_{\mathrm{e}}=\frac{1}{2} \varepsilon_{0} \frac{\pi r^{2}}{d^{2}} V^{2}=\frac{\chi}{2 d} V^{2} . \tag{c2}
\end{equation*}
$$

In order for the disk to be lifted, one requires $F_{\text {net }}>0$ :

$$
\begin{align*}
& \frac{\chi}{2 d} V^{2}-m g>0 .  \tag{c3}\\
& \therefore V_{\mathrm{th}}=\sqrt{\frac{2 m g d}{\chi}} . \tag{c4}
\end{align*}
$$

(d) [2.3 points]

Let $\mathrm{v}_{\mathrm{s}}$ be the steady velocity of the small disk just after its collision with the bottom plate. Then the steady-state kinetic energy $K_{\mathrm{s}}$ of the disk just above the bottom plate is given by

$$
\begin{equation*}
K_{\mathrm{s}}=\frac{1}{2} m \mathrm{v}_{\mathrm{s}}^{2} . \tag{d1}
\end{equation*}
$$

For each round trip, the disk gains electrostatic energy by

$$
\begin{equation*}
\Delta U=2 q \mathrm{~V} . \tag{d2}
\end{equation*}
$$

For each inelastic collision, the disk lose its kinetic energy by

$$
\begin{equation*}
\Delta K_{\text {loss }}=K_{\text {before }}-K_{\text {after }}=\left(1-\eta^{2}\right) K_{\text {before }}=\left(\frac{1}{\eta^{2}}-1\right) K_{\text {affer }} \tag{d3}
\end{equation*}
$$

Since $K_{\mathrm{s}}$ is the energy after the collision at the bottom plate and $\left(K_{\mathrm{s}}+q V-m g d\right)$ is
the energy before the collision at the top plate, the total energy loss during the round trip can be written in terms of $K_{\mathrm{s}}$ :

$$
\begin{equation*}
\Delta K_{\mathrm{tot}}=\left(\frac{1}{\eta^{2}}-1\right) K_{\mathrm{s}}+\left(1-\eta^{2}\right)\left(K_{\mathrm{s}}+q V-m g d\right) . \tag{d4}
\end{equation*}
$$

In its steady state, $\Delta U$ should be compensated by $\Delta K_{\text {tot }}$.

$$
\begin{equation*}
2 q V=\left(\frac{1}{\eta^{2}}-1\right) K_{\mathrm{s}}+\left(1-\eta^{2}\right)\left(K_{\mathrm{s}}+q V-m g d\right) . \tag{d5}
\end{equation*}
$$

Rearranging Eq. (d5), we have

$$
\begin{align*}
K_{\mathrm{s}} & =\frac{\eta^{2}}{1-\eta^{4}}\left[\left(1+\eta^{2}\right) q V+\left(1-\eta^{2}\right) m g d\right] \\
& =\left(\frac{\eta^{2}}{1-\eta^{2}}\right) q V+\left(\frac{\eta^{2}}{1+\eta^{2}}\right) m g d  \tag{d6}\\
& =\frac{1}{2} m \mathrm{v}_{\mathrm{s}}^{2} .
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\mathrm{v}_{\mathrm{s}}=\sqrt{\left(\frac{\eta^{2}}{1-\eta^{2}}\right)\left(\frac{2 \chi V^{2}}{m}\right)+\left(\frac{\eta^{2}}{1+\eta^{2}}\right)(2 g d)} . \tag{d7}
\end{equation*}
$$

Comparing with the form:

$$
\begin{align*}
& \quad \mathrm{v}_{\mathrm{s}}=\sqrt{\alpha V^{2}+\beta}  \tag{d8}\\
& \alpha=\left(\frac{\eta^{2}}{1-\eta^{2}}\right)\left(\frac{2 \chi}{m}\right), \quad \beta=\left(\frac{\eta^{2}}{1+\eta^{2}}\right)(2 g d) . \tag{d9}
\end{align*}
$$

## [An alternative solution:]

Let $\mathrm{v}_{n}$ be the velocity of the small disk just after $n$-th collision with the bottom plate. Then the kinetic energy of the disk just above the bottom plate is given by

$$
\begin{equation*}
K_{n}=\frac{1}{2} m v_{n}^{2} . \tag{d10}
\end{equation*}
$$

When it reaches the top plate, the disk gains energy by the increase of potential energy:

$$
\begin{equation*}
\Delta U_{\text {up }}=q V-m g d \tag{d11}
\end{equation*}
$$

Thus, the kinetic energy just before its collision with the top plate becomes

$$
\begin{equation*}
K_{n-\text { up }}=\frac{1}{2} m \mathrm{v}_{\text {up }}^{2}=K_{n}+\Delta U_{\text {up }} . \tag{d12}
\end{equation*}
$$

Since $\eta=\mathrm{v}_{\text {after }} / \mathrm{v}_{\text {before }}$, the kinetic energy after the collision with the top plate becomes scaled down by a factor of $\eta^{2}$ :

$$
\begin{equation*}
K_{n \text {-up }}^{\prime}=\eta^{2} \cdot K_{n \text {-up }} . \tag{d13}
\end{equation*}
$$

Now the potential energy gain by the downward motion is:

$$
\begin{equation*}
\Delta U_{\mathrm{down}}=q V+m g d \tag{d14}
\end{equation*}
$$

so that the kinetic energy just before it collides with the bottom plate becomes:

$$
\begin{equation*}
K_{n \text {-down }}=K_{n \text {-up }}^{\prime}+\Delta U_{\text {down }} . \tag{d15}
\end{equation*}
$$

Again, due to the loss of energy by the collision with the bottom plate, the kinetic energy after its $(n+1)$-th collision can be obtained by

$$
\begin{align*}
K_{n+1} & =\eta^{2} \cdot K_{n \text {-down }} \\
& =\eta^{2}\left(K_{n \text {-up }}^{\prime}+\Delta U_{\text {down }}\right) \\
& =\eta^{2}\left(\eta^{2}\left(K_{n}+\Delta U_{\text {up }}\right)+\Delta U_{\text {down }}\right)  \tag{d16}\\
& =\eta^{2}\left(\eta^{2}\left(K_{n}+q V-m g d\right)+q V+m g d\right) \\
& =\eta^{4} K_{n}+\eta^{2}\left(1+\eta^{2}\right) q V+\eta^{2}\left(1-\eta^{2}\right) m g d .
\end{align*}
$$

As $n \rightarrow \infty$, we expect the velocity $\mathrm{v}_{n} \rightarrow \mathrm{v}_{\mathrm{s}}$, that is, $K_{n} \rightarrow K_{\mathrm{s}}=\frac{1}{2} m \mathrm{v}_{\mathrm{s}}^{2}$ :

$$
\begin{align*}
K_{\mathrm{s}} & =\frac{1}{1-\eta^{4}}\left[\eta^{2}\left(1+\eta^{2}\right) q V+\eta^{2}\left(1-\eta^{2}\right) m g d\right] \\
& =\left(\frac{\eta^{2}}{1-\eta^{2}}\right) q V+\left(\frac{\eta^{2}}{1+\eta^{2}}\right) m g d  \tag{d17}\\
& =\frac{1}{2} m \mathrm{v}_{\mathrm{s}}^{2}
\end{align*}
$$

(e) $[2.2$ points $]$

The amount of charge carried by the disk during its round trip between the plates is $\Delta Q=2 q$, and the time interval $\Delta t=t_{+}+t_{-}$, where $t_{+}\left(t_{-}\right)$is the time spent during the up- (down-) ward motion respectively.
Here $t_{+}\left(t_{-}\right)$can be determined by

$$
\begin{align*}
& \mathrm{v}_{0+} t_{+}+\frac{1}{2} a_{+} t_{+}^{2}=d \\
& \mathrm{v}_{0-} t_{-}+\frac{1}{2} a_{-} t_{-}^{2}=d \tag{e1}
\end{align*}
$$

where $\mathrm{v}_{0+}\left(\mathrm{v}_{0-}\right)$ is the initial velocity at the bottom (top) plate and $a_{+}\left(a_{-}\right)$is the up-
(down-) ward acceleration respectively.
Since the force acting on the disk is given by

$$
\begin{equation*}
F=m a_{ \pm}=q E \mp m g=\frac{q V}{d} \mp m g \tag{e2}
\end{equation*}
$$

in the limit of $m g d \ll q V, a_{ \pm}$can be approximated by

$$
\begin{equation*}
a_{0}=a_{+}=a_{-} \approx \frac{q V}{m d}, \tag{e3}
\end{equation*}
$$

which implies that the upward and down-ward motion should be symmetric. Thus, Eq.(e1) can be described by a single equation with $t_{0}=t_{+}=t_{-}, \mathrm{v}_{\mathrm{s}}=\mathrm{v}_{0+}=\mathrm{v}_{0 .}$, and $a_{0}=a_{+}=a_{-}$. Moreover, since the speed of the disk just after the collision should be the same for the top- and bottom-plates, one can deduce the relation:

$$
\begin{equation*}
\mathrm{v}_{\mathrm{s}}=\eta\left(\mathrm{v}_{\mathrm{s}}+a_{0} t_{0}\right), \tag{e4}
\end{equation*}
$$

from which we obtain the time interval $\Delta t=2 t_{0}$,

$$
\begin{equation*}
\Delta t=2 t_{0}=2\left(\frac{1-\eta}{\eta}\right) \frac{\mathrm{v}_{\mathrm{s}}}{a_{0}} . \tag{e5}
\end{equation*}
$$

From Eq. (d6), in the limit of $m g d \ll q V$, we have

$$
\begin{equation*}
K_{\mathrm{s}}=\frac{1}{2} m \mathrm{v}_{\mathrm{s}}^{2} \approx\left(\frac{\eta^{2}}{1-\eta^{2}}\right) q V . \tag{e6}
\end{equation*}
$$

By substituting the results of Eqs. (e3) and (e6), we get

$$
\begin{equation*}
\Delta t=2\left(\frac{1-\eta}{\eta}\right) \sqrt{\frac{2 \eta^{2}}{1-\eta^{2}}} \sqrt{\frac{m d^{2}}{q V}}=2 \sqrt{\frac{1-\eta}{1+\eta}} \sqrt{\frac{2 m d^{2}}{\chi V^{2}}} . \tag{e7}
\end{equation*}
$$

Therefore, from $I=\frac{\Delta Q}{\Delta t}=\frac{2 q}{\Delta t}$,

$$
\begin{align*}
& I=\frac{2 q}{\Delta t}=\chi^{V} \sqrt{\frac{1+\eta}{1-\eta}} \sqrt{\frac{\chi^{V^{2}}}{2 m d^{2}}}=\sqrt{\frac{1+\eta}{1-\eta}} \sqrt{\frac{\chi^{3}}{2 m d^{2}}} V^{2} .  \tag{e8}\\
& \therefore \gamma=\sqrt{\frac{1+\eta}{1-\eta}} \sqrt{\frac{\chi^{3}}{2 m d^{2}}} \tag{e9}
\end{align*}
$$

## [Alternative solution \#1:]

Starting from Eq. (e3), we can solve the quadratic equation of Eq. (e1) so that

$$
\begin{equation*}
t_{ \pm}=\frac{\mathrm{v}_{0 \pm}}{a_{0}}\left(\sqrt{1+\frac{2 d a_{0}}{\mathrm{v}_{0 \pm}^{2}}}-1\right) . \tag{e10}
\end{equation*}
$$

When it reaches the steady state, the initial velocities $\mathrm{v}_{0 \pm}$ are given by

$$
\begin{gather*}
\mathrm{v}_{0+}=\mathrm{v}_{\mathrm{s}}  \tag{e11}\\
\mathrm{v}_{0-}=\eta \cdot\left(\mathrm{v}_{\mathrm{s}}+a_{0} t_{+}\right)=\eta \mathrm{v}_{\mathrm{s}} \sqrt{1+\frac{2 d a_{0}}{\mathrm{v}_{\mathrm{s}}^{2}}}, \tag{e12}
\end{gather*}
$$

where $\mathrm{v}_{\mathrm{s}}$ can be rewritten by using the result of Eq. (e6),

$$
\begin{equation*}
\mathrm{v}_{\mathrm{s}}^{2} \approx \alpha V=\left(\frac{\eta^{2}}{1-\eta^{2}}\right) \frac{2 q V}{m}=\left(\frac{\eta^{2}}{1-\eta^{2}}\right) 2 a_{0} d . \tag{e13}
\end{equation*}
$$

As a result, we get $\mathrm{v}_{0-} \cong \eta \mathrm{v}_{\mathrm{s}} \cdot \frac{1}{\eta}=\mathrm{v}_{\mathrm{s}}$ and consequently $t_{ \pm}=\frac{\mathrm{v}_{\mathrm{s}}}{a_{0}}\left(\frac{1}{\eta}-1\right)$, which is equivalent to Eq. (e4).

## [Alternative solution \#2:]

The current $I$ can be obtained from

$$
\begin{equation*}
I=\frac{2 q}{\Delta t}=\frac{2 q \overline{\mathrm{v}}}{d} \tag{e14}
\end{equation*}
$$

where $\overline{\mathrm{v}}$ is an average velocity. Since the up and down motions are symmetric with the same constant acceleration in the limit of $m g d \ll q V$,

$$
\begin{equation*}
\overline{\mathrm{v}}=\frac{1}{2}\left(\mathrm{v}_{\mathrm{s}}+\frac{\mathrm{v}_{\mathrm{s}}}{\eta}\right) . \tag{e15}
\end{equation*}
$$

Thus, we have

$$
\begin{equation*}
I=\frac{q}{2 d}\left(1+\frac{1}{\eta}\right) \mathrm{v}_{\mathrm{s}} . \tag{e16}
\end{equation*}
$$

Inserting the expression (Eq. (e15)) of $\mathrm{v}_{\mathrm{s}}$ into Eq. (e16), one obtains an expression identical to Eq. (e8).
(f) [3 points]

The disk will lose its kinetic energy and eventually cease to move when the disk can not reach the top plate. In other words, the threshold voltage $V_{c}$ can be determined from the condition that the velocity $\mathrm{v}_{0 .}$ of the disk at the top plate is zero, i.e., $\mathrm{v}_{0-}=0$.

In order for the disk to have $\mathrm{v}_{0-}=0$ at the top plate, the kinetic energy $\bar{K}_{\mathrm{s}}$ at the
top plate should satisfy the relation:

$$
\begin{equation*}
\bar{K}_{\mathrm{s}}=K_{\mathrm{s}}+q V_{c}-m g d=0, \tag{f1}
\end{equation*}
$$

where $K_{\mathrm{s}}$ is the steady-state kinetic energy at the bottom plate after the collision. Therefore, we have

$$
\begin{equation*}
\left(\frac{\eta^{2}}{1-\eta^{2}}\right) q V_{c}+\left(\frac{\eta^{2}}{1+\eta^{2}}\right) m g d+q V_{c}-m g d=0 \tag{f2}
\end{equation*}
$$

or equivalently,

$$
\begin{align*}
& \left(1+\eta^{2}\right) q V_{c}-\left(1-\eta^{2}\right) m g d=0 .  \tag{f3}\\
& \therefore \quad q V_{\mathrm{c}}=\frac{1-\eta^{2}}{1+\eta^{2}} m g d \tag{f4}
\end{align*}
$$

From the relation $q=\chi V_{c}$,

$$
\begin{equation*}
\therefore V_{\mathrm{c}}=\sqrt{\frac{1-\eta^{2}}{1+\eta^{2}}} \sqrt{\frac{m g d}{\chi}} . \tag{f5}
\end{equation*}
$$

In comparison with the threshold voltage $V_{\text {th }}$ of Eq. (c4), we can rewrite Eq. (f5) by

$$
\begin{equation*}
V_{c}=z_{c} V_{\mathrm{th}} \tag{f6}
\end{equation*}
$$

where $z_{c}$ should be used in the plot of $I$ vs. $\left(V / V_{\text {th }}\right)$ and

$$
\begin{equation*}
z_{c}=\sqrt{\frac{1-\eta^{2}}{2\left(1+\eta^{2}\right)}} . \tag{f7}
\end{equation*}
$$

[Note that an alternative derivation of Eq. (fl) is possible if one applies the energy compensation condition of Eq. (d5) or the recursion relation of Eq. (d17) at the top plate instead of the bottom plate.]

Now we can setup equations to determine the time interval $\Delta t=t_{-}+t_{+}$:

$$
\begin{align*}
& \mathrm{v}_{0-} t_{-}+\frac{1}{2} a_{-} t_{-}^{2}=d  \tag{f8}\\
& \mathrm{v}_{0_{+}} t_{+}+\frac{1}{2} a_{+} t_{+}^{2}=d \tag{f9}
\end{align*}
$$

where the accelerations are given by

$$
\begin{equation*}
a_{+}=\frac{q V_{\mathrm{c}}}{m d}-g=\left[\frac{1-\eta^{2}}{1+\eta^{2}}-1\right] g=\left(\frac{-2 \eta^{2}}{1+\eta^{2}}\right) g \tag{f10}
\end{equation*}
$$

$$
\begin{gather*}
a_{-}=\frac{q V_{\mathrm{c}}}{m d}+g=\left[\frac{1-\eta^{2}}{1+\eta^{2}}+1\right] g=\left(\frac{2}{1+\eta^{2}}\right) g  \tag{fl1}\\
\frac{a_{+}}{a_{-}}=-\eta^{2} \tag{f12}
\end{gather*}
$$

Since $\mathrm{v}_{0-}=0$, we have $\mathrm{v}_{0+}=\eta\left(a_{-} t_{-}\right)$and $t_{-}^{2}=2 d / a_{-}$.

$$
\begin{equation*}
t_{-}=\sqrt{\frac{2 d}{a_{-}}}=\sqrt{\left(1+\eta^{2}\right)\left(\frac{d}{g}\right)}, \tag{f13}
\end{equation*}
$$

By using $\mathrm{v}_{0_{+}}^{2}=\eta^{2}\left(2 d a_{-}\right)=-2 d a_{+}$, we can solve the quadratic equation of Eq. (f9):

$$
\begin{gather*}
t_{+}=\frac{\mathrm{v}_{0+}}{a_{+}}\left(\sqrt{1+\frac{2 d a_{+}}{\mathrm{v}_{0+}^{2}}}-1\right)=-\frac{\mathrm{v}_{0+}}{a_{+}}=\sqrt{\frac{2 d}{a_{+} \mid}}=\sqrt{\left(\frac{1+\eta^{2}}{\eta^{2}}\right)\left(\frac{d}{g}\right)}=\frac{t_{-}}{\eta} .  \tag{f14}\\
\therefore \Delta t=t_{-}+t_{+}=\left(1+\frac{1}{\eta}\right) \sqrt{\left(1+\eta^{2}\right)\left(\frac{d}{g}\right)}  \tag{f15}\\
I_{\mathrm{c}}=\frac{\Delta Q_{\mathrm{c}}}{\Delta t}=\frac{2 q}{\Delta t}=\frac{2 \chi V_{\mathrm{c}}}{\Delta t}=\frac{2 \eta \sqrt{1-\eta^{2}}}{(1+\eta)\left(1+\eta^{2}\right)} g \sqrt{m \chi} . \tag{f16}
\end{gather*}
$$



## [A more elaborate Solution:]

One may find a general solution for an arbitrary value of $V$. By solving the quadratic equations of Eqs. (f8) and (f9), we have

$$
\begin{equation*}
t_{ \pm}=\frac{\mathrm{v}_{0 \pm}}{a_{ \pm}}\left[-1+\sqrt{1+\frac{2 d a_{ \pm}}{\mathrm{v}_{0 \pm}^{2}}}\right] . \tag{f17}
\end{equation*}
$$

(It is noted that one has to keep the smaller positive root.)

To simplify the notation, we introduce a few variables:
(i) $y=\frac{V}{V_{\text {th }}}$ where $V_{\text {th }}=\sqrt{\frac{2 m g d}{\chi}}$,
(ii) $z_{c}=\sqrt{\frac{1-\eta^{2}}{2\left(1+\eta^{2}\right)}}$, which is defined in Eq. (f7),
(iii) $w_{0}=2 \eta \sqrt{\frac{g d}{1-\eta^{2}}}$ and $w_{1}=2 \sqrt{\frac{d}{\left(1-\eta^{2}\right) g}}$,

In terms of $y, w$, and $z_{c}$,

$$
\begin{align*}
& a_{+}=\frac{q V}{m d}-g=g\left(2 y^{2}-1\right)  \tag{f18}\\
& a_{-}=\frac{q V}{m d}+g=g\left(2 y^{2}+1\right)  \tag{f19}\\
& \mathrm{v}_{0+}=\mathrm{v}_{\mathrm{s}}=w_{0} \sqrt{y^{2}+z_{c}^{2}}  \tag{f20}\\
& \mathrm{v}_{0-}=\eta\left(\mathrm{v}_{\mathrm{s}}+a_{+} t_{+}\right)=w_{0} \sqrt{y^{2}-z_{c}^{2}}  \tag{f21}\\
& t_{+}=w_{1} \frac{\sqrt{y^{2}-z_{c}^{2}}-\eta \sqrt{y^{2}+z_{c}^{2}}}{2 y^{2}-1}  \tag{f22}\\
& t_{-}=w_{1} \frac{\sqrt{y^{2}+z_{c}^{2}}-\eta \sqrt{y^{2}-z_{c}^{2}}}{2 y^{2}+1} \tag{f21}
\end{align*}
$$

$$
\begin{equation*}
I=\frac{\Delta Q}{\Delta t}=\frac{2 q}{t_{+}+t_{-}}=\left(2 \chi V_{\mathrm{th}}\right) \frac{y}{\Delta t}=\frac{\sqrt{8 m g d \chi}}{w_{1}} F(y) \tag{f22}
\end{equation*}
$$

where

$$
\begin{equation*}
F(y)=y\left\{\frac{\sqrt{y^{2}-z_{c}^{2}}-\eta \sqrt{y^{2}+z_{c}^{2}}}{2 y^{2}-1}+\frac{\sqrt{y^{2}+z_{c}^{2}}-\eta \sqrt{y^{2}-z_{c}^{2}}}{2 y^{2}+1}\right\}^{-1} \tag{f23}
\end{equation*}
$$



## 3. Mark Distribution

| No. | Total Pt. | Partial Pt. | Contents |  |
| :---: | :---: | :---: | :---: | :---: |
| (a) | 1.2 | 0.3 | Gauss law, or a formula for the capacitance of a parallel plate |  |
|  |  | 0.5 | Total energy of a capacitor at V | $E^{\prime}=$ electrical field by the other plate |
|  |  | 0.4 | Force from the energy expression | $F=Q E^{\prime}$ |
| (b) | 0.8 | 0.3 | Gauss law $\quad$ Use of area ratio and result of (a) |  |
|  |  | 0.5 | Correct answer |  |
| (c) | 0.5 | 0.1 | Correct lift-up condition with force balance |  |
|  |  | 0.2 | Use of area ratio and result of (a) |  |
|  |  | 0.2 | Correct answer |  |
| (d) | 2.3 | 0.5 | Energy conservation and the work done by the field |  |
|  |  | 0.5 | Loss of energy due to collisions |  |
|  |  | 0.8 | Condition for the steady state: energy balance equation (loss $=$ gain) | Condition for the steady state: recursion relation |
|  |  | 0.5 | Correct answer |  |
| (e) | 2.2 | 0.2 | $\Delta Q=2 q$ per trip |  |
|  |  | 0.5 | Acceleration $a_{ \pm}$in the limit of $q V \gg m g d ; a_{+}=a_{-}$by symmetry |  |
|  |  | 0.3 | Kinetic equations for $d, \mathrm{v}, \quad$ By using the symmetry, derive |  |
|  |  | 0.4 | $\begin{aligned} & \text { Expression of } \\ & \text { its steady state }\end{aligned}$$\mathrm{V}_{0 \pm}$ and $t_{ \pm}$in the relation (e4) |  |
|  |  | 0.4 | Solutions of $t_{ \pm}$in approximation |  |
|  |  | 0.4 | Correct answer |  |
| (f) | 3.0 | 0.5 | Condition for $V_{c} ; K_{\text {up }}=0$ or $\mathrm{v}_{\mathrm{s}, \text { यp }}=0$ | Using (d8), Recursion relations |
|  |  | 0.3 | energy balance equation |  |
|  |  | 0.3 | Correct answer of $V_{c}$ |  |
|  |  | 0.7 | Kinetic equations for $\Delta t$ |  |
|  |  | 0.3 | Correct answer of $I_{c}$ |  |
|  |  | 0.9 | Distinction between $V_{\text {th }}$ and $V_{c}$, the asymptotic behavior $\quad I=\gamma V^{c}{ }^{2}$ in plots |  |
| Total | 10 |  |  |  |

## Theoretical Question 2: Rising Balloon

1. Answers
(a) $F_{B}=M_{A} n g \frac{P}{P+\Delta P}$
(b) $\gamma=\frac{\rho_{0} z_{0} g}{P_{0}}=5.5$
(c) $\Delta P=\frac{4 \kappa R T}{r_{0}}\left(\frac{1}{\lambda}-\frac{1}{\lambda^{7}}\right)$

(d) $a=0.110$
(e) $z_{f}=11 \mathrm{~km}, \quad \lambda_{f}=2.1$.

## 2. Solutions

## [Part A]

(a) $[1.5$ points]

Using the ideal gas equation of state, the volume of the helium gas of $n$ moles at pressure $P+\Delta P$ and temperature $T$ is

$$
\begin{equation*}
V=n R T /(P+\Delta P) \tag{a1}
\end{equation*}
$$

while the volume of $n^{\prime}$ moles of air gas at pressure $P$ and temperature $T$ is

$$
\begin{equation*}
V=n^{\prime} R T / P . \tag{a2}
\end{equation*}
$$

Thus the balloon displaces $n^{\prime}=n \frac{P}{P+\Delta P}$ moles of air whose weight is $M_{A} n^{\prime} g$.
This displaced air weight is the buoyant force, i.e.,

$$
\begin{equation*}
F_{B}=M_{A} n g \frac{P}{P+\Delta P} . \tag{a3}
\end{equation*}
$$

(Partial credits for subtracting the gas weight.)
(b) $[2$ points $]$

The pressure difference arising from a height difference of $z$ is $-\rho g z$ when the air density $\rho$ is a constant. When it varies as a function of the height, we have

$$
\begin{equation*}
\frac{d P}{d z}=-\rho g=-\frac{\rho_{0} T_{0}}{P_{0}} \frac{P}{T} g \tag{b1}
\end{equation*}
$$

where the ideal gas law $\rho T / P=$ constant is used. Inserting Eq. (2.1) and $T / T_{0}=1-z / z_{0}$ on both sides of Eq. (b1), and comparing the two, one gets

$$
\begin{equation*}
\gamma=\frac{\rho_{0} z_{0} g}{P_{0}}=\frac{1.16 \times 4.9 \times 10^{4} \times 9.8}{1.01 \times 10^{5}}=5.52 \tag{b2}
\end{equation*}
$$

The required numerical value is 5.5 .

## [Part B]

(c) [2 points]

The work needed to increase the radius from $r$ to $r+d r$ under the pressure difference $\Delta P$ is

$$
\begin{equation*}
d W=4 \pi r^{2} \Delta P d r, \tag{c1}
\end{equation*}
$$

while the increase of the elastic energy for the same change of $r$ is

$$
\begin{equation*}
d W=\left(\frac{d U}{d r}\right) d r=4 \pi \kappa R T\left(4 r-4 \frac{r_{0}^{6}}{r^{5}}\right) d r . \tag{c2}
\end{equation*}
$$

Equating the two expressions of $d W$, one gets

$$
\begin{equation*}
\Delta P=4 \kappa R T\left(\frac{1}{r}-\frac{r_{0}{ }^{6}}{r^{7}}\right)=\frac{4 \kappa R T}{r_{0}}\left(\frac{1}{\lambda}-\frac{1}{\lambda^{7}}\right) . \tag{c3}
\end{equation*}
$$

This is the required answer.
The graph as a function of $\lambda(>1)$ increases sharply initially, has a maximum at $\lambda=7^{1 / 6}$ $=1.38$, and decreases as $\lambda^{-1}$ for large $\lambda$. The plot of $\Delta P /\left(4 \kappa R T / r_{0}\right)$ is given below.

(d) $[1.5$ points $]$

From the ideal gas law,

$$
\begin{equation*}
P_{0} V_{0}=n_{0} R T_{0} \tag{d1}
\end{equation*}
$$

where $V_{0}$ is the unstretched volume.
At volume $V=\lambda^{3} V_{0}$ containing $n$ moles, the ideal gas law applied to the gas inside at $T=T_{0}$ gives the inside pressure $P_{\text {in }}$ as

$$
\begin{equation*}
P_{\text {in }}=n R T_{0} / V=\frac{n}{n_{0} \lambda^{3}} P_{0} . \tag{d2}
\end{equation*}
$$

On the other hand, the result of (c) at $T=T_{0}$ gives

$$
\begin{equation*}
P_{\mathrm{in}}=P_{0}+\Delta P=P_{0}+\frac{4 \kappa R T_{0}}{r_{0}}\left(\frac{1}{\lambda}-\frac{1}{\lambda^{7}}\right)=\left(1+a\left(\frac{1}{\lambda}-\frac{1}{\lambda^{7}}\right)\right) P_{0} . \tag{d3}
\end{equation*}
$$

Equating (d2) and (d3) to solve for $a$,

$$
\begin{equation*}
a=\frac{n /\left(n_{0} \lambda^{3}\right)-1}{\lambda^{-1}-\lambda^{-7}} . \tag{d5}
\end{equation*}
$$

Inserting $n / n_{0}=3.6$ and $\lambda=1.5$ here, $a=0.110$.

## [Part C]

(e) $[3$ points]

The buoyant force derived in problem (a) should balance the total mass of $M_{\mathrm{T}}=1.12 \mathrm{~kg}$. Thus, from Eq. (a3), at the weight balance,

$$
\begin{equation*}
\frac{P}{P+\Delta P}=\frac{M_{\mathrm{T}}}{M_{A} n} . \tag{e1}
\end{equation*}
$$

On the other hand, applying again the ideal gas law to the helium gas inside of volume $V=\frac{4}{3} \pi r^{3}=\lambda^{3} \frac{4}{3} \pi r_{0}{ }^{3}=\lambda^{3} V_{0}$, for arbitrary ambient $P$ and $T$, one has

$$
\begin{equation*}
(P+\Delta P) \lambda^{3}=\frac{n R T}{V_{0}}=P_{0} \frac{T}{T_{0}} \frac{n}{n_{0}} \tag{e2}
\end{equation*}
$$

for $n$ moles of helium. Eqs. (c3), (e1), and (e2) determine the three unknowns $P$, $\Delta P$, and $\lambda$ as a function of $T$ and other parameters. Using Eq. (e2) in Eq. (e1), one has an alternative condition for the weight balance as

$$
\begin{equation*}
\frac{P}{P_{0}} \frac{T_{0}}{T} \lambda^{3}=\frac{M_{\mathrm{T}}}{M_{A} n_{0}} . \tag{e3}
\end{equation*}
$$

Next using (c3) for $\Delta P$ in (e2), one has

$$
P \lambda^{3}+\frac{4 \kappa R T}{r_{0}} \lambda^{2}\left(1-\lambda^{-6}\right)=P_{0} \frac{T}{T_{0}} \frac{n}{n_{0}}
$$

or, rearranging it,

$$
\begin{equation*}
\frac{P}{P_{0}} \frac{T_{0}}{T} \lambda^{3}=\frac{n}{n_{0}}-a \lambda^{2}\left(1-\lambda^{-6}\right), \tag{e4}
\end{equation*}
$$

where the definition of $a$ has been used again.
Equating the right hand sides of Eqs. (e3) and (e4), one has the equation for $\lambda$ as

$$
\begin{equation*}
\lambda^{2}\left(1-\lambda^{-6}\right)=\frac{1}{a n_{0}}\left(n-\frac{M_{\mathrm{T}}}{M_{A}}\right)=4.54 \tag{e5}
\end{equation*}
$$

The solution for $\lambda$ can be obtained by

$$
\begin{equation*}
\lambda^{2} \approx 4.54 /\left(1-4.54^{-3}\right) \approx 4.54: \quad \lambda_{f} \cong 2.13 . \tag{e6}
\end{equation*}
$$

To find the height, replace $\left(P / P_{0}\right) /\left(T / T_{0}\right)$ on the left hand side of Eq. (e3) as a function of the height given in (b) as

$$
\begin{equation*}
\frac{P}{P_{0}} \frac{T_{0}}{T} \lambda^{3}=\left(1-z_{f} / z_{0}\right)^{\gamma-1} \lambda_{f}^{3}=\frac{M_{\mathrm{T}}}{M_{A} n_{0}}=3.10 . \tag{e7}
\end{equation*}
$$

Solution of Eq. (e7) for $z_{f}$ with $\lambda_{f}=2.13$ and $\gamma-1=4.5$ is

$$
\begin{equation*}
z_{f}=49 \times\left(1-\left(3.10 / 2.13^{3}\right)^{1 / 4.5}\right)=10.9(\mathrm{~km}) . \tag{e8}
\end{equation*}
$$

The required answers are $\lambda_{f}=2.1$, and $z_{f}=11 \mathrm{~km}$.

## 3. Mark Distribution

| No. | Total Pt. | Partial Pt. | Contents |
| :---: | :---: | :---: | :---: |
| (a) | 1.5 | 0.5 | Archimedes' principle |
|  |  | 0.5 | Ideal gas law applied correctly |
|  |  | 0.5 | Correct answer (partial credits 0.3 for subtracting He weight) |
| (b) | 2.0 | 0.8 | Relation of pressure difference to air density |
|  |  | 0.5 | Application of ideal gas law to convert the density into pressure |
|  |  | 0.5 | Correct formula for $\gamma$ |
|  |  | 0.2 | Correct number in answer |
| (c) | 2.0 | 0.7 | Relation of mechanical work to elastic energy change |
|  |  | 0.3 | Relation of pressure to force |
|  |  | 0.5 | Correct answer in formula |
|  |  | 0.5 | Correct sketch of the curve |
| (d) | 1.5 | 0.3 | Use of ideal gas law for the increased pressure inside |
|  |  | 0.4 | Expression of inside pressure in terms of $a$ at the given conditions |
|  |  | 0.5 | Formula or correct expression for $a$ |
|  |  | 0.3 | Correct answer |
| (e) | 3.0 | 0.3 | Use of force balance as one condition to determine unknowns |
|  |  | 0.3 | Ideal gas law applied to the gas as an independent condition to determine unknowns |
|  |  | 0.5 | The condition to determine $\lambda_{f}$ numerically |
|  |  | 0.7 | Correct answer for $\lambda_{f}$ |
|  |  | 0.5 | The relation of $z_{f}$ versus $\lambda_{f}$ |
|  |  | 0.7 | Correct answer for $z_{f}$ |
| Total | 10 |  |  |

## Theoretical Question 3: Scanning Probe Microscope

## 1. Answers

(a) $A=\frac{F_{0}}{\sqrt{m^{2}\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+b^{2} \omega^{2}}}$ and $\tan \phi=\frac{b \omega_{0}}{m\left(\omega_{0}^{2}-\omega^{2}\right)}$. At $\omega=\omega_{0}, \quad A=\frac{F_{0}}{b \omega_{0}}$ and $\phi=\frac{\pi}{2}$.
(b) A non-vanishing dc component exists only when $\omega=\omega_{i}$.

In this case the amplitude of the dc signal will be $\frac{1}{2} V_{i 0} V_{R 0} \cos \phi_{i}$.
(c) $\frac{c_{1} c_{2}}{2} \frac{V_{R 0}^{2}}{b \omega_{0}}$ at the resonance frequency $\omega_{0}$.
(d) $\Delta m=1.7 \times 10^{-18} \mathrm{~kg}$.
(e) $\omega_{0}^{\prime}=\omega_{0}\left(1-\frac{c_{3}}{m \omega_{0}^{2}}\right)^{1 / 2}$.
(f) $d_{0}=\left(k_{e} \frac{q Q}{m \omega_{0} \Delta \omega_{0}}\right)^{1 / 3}$

$$
d_{0}=41 \mathrm{~nm} .
$$

## 2. Solutions

(a) $[1.5$ points $]$

Substituting $\quad z(t)=A \sin (\omega t-\phi)$ in the equation $m \frac{d^{2} z}{d t^{2}}+b \frac{d z}{d t}+m \omega_{0}^{2} z=F_{0} \sin \omega t$ yields,

$$
\begin{equation*}
-m \omega^{2} \sin (\omega t-\phi)+b \omega \cos (\omega t-\phi)+m \omega_{0}{ }^{2} \sin (\omega t-\phi)=\frac{F_{0}}{A} \sin \omega t \tag{a1}
\end{equation*}
$$

Collecting terms proportional to $\sin \omega t$ and $\cos \omega t$, one obtains

$$
\left\{m\left(\omega_{0}^{2}-\omega^{2}\right) \cos \phi+b \omega \sin \phi-\frac{F_{0}}{A}\right\} \sin \omega t+\left\{-m\left(\omega_{0}^{2}-\omega^{2}\right) \sin \phi+b \omega \cos \phi\right\} \cos \omega t=0(\mathrm{a} 2)
$$

Zeroing the each curly square bracket produces

$$
\begin{gather*}
\tan \phi=\frac{b \omega}{m\left(\omega_{0}^{2}-\omega^{2}\right)},  \tag{a3}\\
A=\frac{F_{0}}{\sqrt{m^{2}\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+b^{2} \omega^{2}}} . \tag{a4}
\end{gather*}
$$

At $\omega=\omega_{0}$,

$$
\begin{equation*}
A=\frac{F_{0}}{b \omega_{0}} \text { and } \phi=\frac{\pi}{2} . \tag{a5}
\end{equation*}
$$

(b) $[1$ point $]$

The multiplied signal is

$$
\begin{align*}
& V_{i 0} \sin \left(\omega_{i} t-\phi_{i}\right) V_{R 0} \sin (\omega t) \\
& =\frac{1}{2} V_{i 0} V_{R 0}\left[\cos \left\{\left(\omega_{i}-\omega\right) t-\phi_{i}\right\}-\cos \left\{\left(\omega_{i}+\omega\right) t-\phi_{i}\right\}\right] \tag{b1}
\end{align*}
$$

A non-vanishing dc component exists only when $\omega=\omega_{i}$. In this case the amplitude of the dc signal will be

$$
\begin{equation*}
\frac{1}{2} V_{i 0} V_{R 0} \cos \phi_{i} . \tag{b2}
\end{equation*}
$$

(c) $[1.5$ points $]$

Since the lock-in amplifier measures the ac signal of the same frequency with its reference signal, the frequency of the piezoelectric tube oscillation, the frequency of the
cantilever, and the frequency of the photodiode detector should be same. The magnitude of the input signal at the resonance is

$$
\begin{equation*}
V_{i 0}=c_{2} \frac{F_{0}}{b \omega_{0}}=\frac{c_{1} c_{2} V_{R 0}}{b \omega_{0}} . \tag{c1}
\end{equation*}
$$

Then, since the phase of the input signal is $-\frac{\pi}{2}+\frac{\pi}{2}=0$ at the resonance, $\phi_{i}=0$ and the lock-in amplifier signal is

$$
\begin{equation*}
\frac{1}{2} V_{i 0} V_{R 0} \cos 0=\frac{c_{1} c_{2}}{2} \frac{V_{R 0}^{2}}{b \omega_{0}} . \tag{c2}
\end{equation*}
$$

(d) $[2$ points]

The original resonance frequency $\omega_{0}=\sqrt{\frac{k}{m}}$ is shifted to

$$
\begin{equation*}
\sqrt{\frac{k}{m+\Delta m}}=\sqrt{\frac{k}{m}}\left(1+\frac{\Delta m}{m}\right)^{-\frac{1}{2}} \cong \sqrt{\frac{k}{m}}\left(1-\frac{1}{2} \frac{\Delta m}{m}\right)=\omega_{0}\left(1-\frac{1}{2} \frac{\Delta m}{m}\right) . \tag{d1}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\Delta \omega_{0}=-\frac{1}{2} \omega_{0} \frac{\Delta m}{m} . \tag{d2}
\end{equation*}
$$

Near the resonance, by substituting $\phi \rightarrow \frac{\pi}{2}+\Delta \phi$ and $\omega_{0} \rightarrow \omega_{0}+\Delta \omega_{0}$ in Eq. (a3), the change of the phase due to the small change of $\omega_{0}$ (not the change of $\omega$ ) is

$$
\begin{equation*}
\tan \left(\frac{\pi}{2}+\Delta \phi\right)=-\frac{1}{\tan \Delta \phi}=\frac{b}{2 m \Delta \omega_{0}} . \tag{d3}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\Delta \phi \approx \tan \Delta \phi=-\frac{2 m \Delta \omega_{0}}{b} . \tag{d4}
\end{equation*}
$$

From Eqs. (d2) and (d4),

$$
\begin{equation*}
\Delta m=\frac{b}{\omega_{0}} \Delta \phi=\frac{10^{3} \cdot 10^{-12}}{10^{6}} \frac{\pi}{1800}=\frac{\pi}{1.8} 10^{-18}=1.7 \times 10^{-18} \mathrm{~kg} . \tag{d5}
\end{equation*}
$$

(e) $[1.5$ points $]$

In the presence of interaction, the equation of motion near the new equilibrium position $h_{0}$ becomes

$$
\begin{equation*}
m \frac{d^{2} z}{d t^{2}}+b \frac{d z}{d t}+m \omega_{0}^{2} z-c_{3} z=F_{0} \sin \omega t \tag{el}
\end{equation*}
$$

where we used $f(h) \approx f\left(h_{0}\right)+c_{3} z$ with $z=h-h_{0}$ being the displacement from the new equilibrium position $h_{0}$. Note that the constant term $f\left(h_{0}\right)$ is cancelled at the new equilibrium position.
Thus the original resonance frequency $\omega_{0}=\sqrt{\frac{k}{m}}$ will be shifted to

$$
\begin{equation*}
\omega_{0}^{\prime}=\sqrt{\frac{k-c_{3}}{m}}=\sqrt{\frac{m \omega_{0}^{2}-c_{3}}{m}}=\omega_{0} \sqrt{1-\frac{c_{3}}{m \omega_{0}^{2}}} . \tag{e3}
\end{equation*}
$$

Hence the resonance frequency shift is given by

$$
\begin{equation*}
\Delta \omega_{0}=\omega_{0}\left[\sqrt{1-\frac{c_{3}}{m \omega_{0}^{2}}}-1\right] . \tag{e4}
\end{equation*}
$$

## (f) $[2.5$ points]

The maximum shift occurs when the cantilever is on top of the charge, where the interacting force is given by

$$
\begin{equation*}
f(h)=k_{e} \frac{q Q}{h^{2}} . \tag{fl}
\end{equation*}
$$

From this,

$$
\begin{equation*}
c_{3}=\left.\frac{d f}{d h}\right|_{h=d_{0}}=-2 k_{e} \frac{q Q}{d_{0}^{3}} . \tag{f2}
\end{equation*}
$$

Since $\Delta \omega_{0} \ll \omega_{0}$, we can approximate Eq. (e4) as

$$
\begin{equation*}
\Delta \omega_{0} \approx-\frac{c_{3}}{2 m \omega_{0}} . \tag{f3}
\end{equation*}
$$

From Eqs. (f2) and (f3), we have

$$
\begin{equation*}
\Delta \omega_{0}=-\frac{1}{2 m \omega_{0}}\left(-2 k_{e} \frac{q Q}{d_{0}^{3}}\right)=k_{e} \frac{q Q}{m \omega_{0} d_{0}^{3}} . \tag{f4}
\end{equation*}
$$

Here $q=e=-1.6 \times 10^{-19}$ Coulomb and $Q=6 e=-9.6 \times 10^{-19}$ Coulomb. Using the values provided,

$$
\begin{equation*}
d_{0}=\left(k_{e} \frac{q Q}{m \omega_{0} \Delta \omega_{0}}\right)^{1 / 3}=4.1 \times 10^{-8} \mathrm{~m}=41 \mathrm{~nm} . \tag{f5}
\end{equation*}
$$

Thus the trapped electron is 41 nm from the cantilever.

## 3. Mark Distribution

| No. | Total <br> Pt. | $\begin{gathered} \text { Partial } \\ \text { Pt. } \end{gathered}$ | Contents |
| :---: | :---: | :---: | :---: |
| (a) | 1.5 | 0.7 | Equations for $A$ and $\phi$ (substitution and manipulation) |
|  |  | 0.4 | Correct answers for $A$ and $\phi$ |
|  |  | 0.4 | $A$ and $\phi$ at $\omega_{0}$ |
| (b) | 1.0 | 0.4 | Equation for the multiplied signal |
|  |  | 0.3 | Condition for the non-vanishing dc output |
|  |  | 0.3 | Correct answer for the dc output |
| (c) | 1.5 | 0.6 | Relation between $V_{i}$ and $V_{R}$ |
|  |  | 0.4 | Condition for the maximum dc output |
|  |  | 0.5 | Correct answer for the magnitude of dc output |
| (d) | 2.0 | 0.5 | Relation between $\Delta m$ and $\Delta \omega_{0}$ |
|  |  | 1.0 | Relations between $\Delta \omega_{0}($ or $\Delta m)$ and $\Delta \phi$ |
|  |  | 0.5 | Correct answer (Partial credit of 0.2 for the wrong sign.) |
| (e) | 1.5 | 1.0 | Modification of the equation with $f(h)$ and use of a proper approximation for the equation |
|  |  | 0.5 | Correct answer |
| (f) | 2.5 | 0.5 | Use of a correct formula of Coulomb force |
|  |  | 0.3 | Evaluation of $c_{3}$ |
|  |  | 0.6 | Use of the result in (e) for either $\Delta \omega_{0}$ or $\omega_{0}^{\prime 2}-\omega_{0}^{2}$ |
|  |  | 0.6 | Expression for $d_{0}$ |
|  |  | 0.5 | Correct answer |
| Total | 10 |  |  |

## Solutions

## PART-A Product of the mass and the position of the ball $(m \times l)$ (4.0 points)

1. Suggest and justify, by using equations, a method allowing to obtain $m \times l$. (2.0 points)

$$
m \times l=(M+m) \times l_{\mathrm{cm}}
$$

(Explanation) The lever rule is applied to the Mechanical "Black Box", shown in Fig. A-1, once the position of the center of mass of the whole system is found.


Fig. A-1 Experimental setup
2. Experimentally determine the value of $m \times l$. ( 2.0 points)

$$
m \times l=2.96 \times 10^{-3} \mathrm{~kg} \cdot \mathrm{~m}
$$

(Explanation) The measured quantities are

$$
M+m=(1.411 \pm 0.0005) \times 10^{-1} \mathrm{~kg}
$$

and

$$
l_{\mathrm{cm}}=(2.1 \pm 0.06) \times 10^{-2} \mathrm{~m} \quad \text { or } \quad 21 \pm 0.6 \mathrm{~mm} .
$$

Therefore

$$
\begin{aligned}
m \times l & =(M+m) \times l_{\mathrm{cm}} \\
& =(1.411 \pm 0.0005) \times 10^{-1} \mathrm{~kg} \times(2.1 \pm 0.06) \times 10^{-2} \mathrm{~m} \\
& =(2.96 \pm 0.08) \times 10^{-3} \mathrm{~kg} \cdot \mathrm{~m}
\end{aligned}
$$

## PART-B The mass $m$ of the ball ( 10.0 points)

1. Measure $v$ for various values of $h$. Plot the data on a graph paper in a form that is suitable to find the value of $m$. Identify the slow rotation region and the fast rotation region on the graph. (4.0 points)
2. Show from your measurements that $h=C v^{2}$ in the slow rotation region, and $h=$ $A v^{2}+B$ in the fast rotation region. (1.0 points)


Fig. B-1 Experimental data
(Explanation) The measured data are

|  | $h_{1}\left(\times 10^{-2} \mathrm{~m}\right)^{\mathrm{a})}$ | $\Delta t(\mathrm{~ms})$ | $h\left(\times 10^{-2} \mathrm{~m}\right)^{\mathrm{b})}$ | $v\left(\times 10^{-2} \mathrm{~m} / \mathrm{s}\right)^{\mathrm{c})}$ | $v^{2}\left(\times 10^{-4} \mathrm{~m}^{2} / \mathrm{s}^{2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $25.5 \pm 0.1$ | $269.4 \pm 0.05$ | $1.8 \pm 0.1$ | $8.75 \pm 0.02$ | $76.6 \pm 0.2$ |
| 2 | $26.5 \pm 0.1$ | $235.7 \pm 0.05$ | $2.8 \pm 0.1$ | $11.12 \pm 0.02$ | $123.7 \pm 0.3$ |
| 3 | $27.5 \pm 0.1$ | $197.9 \pm 0.05$ | $3.8 \pm 0.1$ | $13.24 \pm 0.03$ | $175.3 \pm 0.6$ |
| 4 | $28.5 \pm 0.1$ | $176.0 \pm 0.05$ | $4.8 \pm 0.1$ | $14.89 \pm 0.03$ | $221.7 \pm 0.6$ |
| 5 | $29.5 \pm 0.1$ | $161.8 \pm 0.05$ | $5.8 \pm 0.1$ | $16.19 \pm 0.03$ | $262.1 \pm 0.7$ |
| 6 | $30.5 \pm 0.1$ | $151.4 \pm 0.05$ | $6.8 \pm 0.1$ | $17.31 \pm 0.03$ | $299.6 \pm 0.7$ |
| 7 | $31.5 \pm 0.1$ | $141.8 \pm 0.05$ | $7.8 \pm 0.1$ | $18.48 \pm 0.04$ | $342 \pm 1$ |
| 8 | $32.5 \pm 0.1$ | $142.9 \pm 0.05$ | $8.8 \pm 0.1$ | $18.33 \pm 0.04$ | $336 \pm 1$ |


| 9 | $33.5 \pm 0.1$ | $141.4 \pm 0.05$ | $9.8 \pm 0.1$ | $18.53 \pm 0.04$ | $343 \pm 1$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | $34.5 \pm 0.1$ | $142.2 \pm 0.05$ | $10.8 \pm 0.1$ | $18.42 \pm 0.04$ | $339 \pm 1$ |
| 11 | $35.5 \pm 0.1$ | $145.4 \pm 0.05$ | $11.8 \pm 0.1$ | $18.02 \pm 0.04$ | $325 \pm 1$ |
| 12 | $36.5 \pm 0.1$ | $147.8 \pm 0.05$ | $12.8 \pm 0.1$ | $17.73 \pm 0.04$ | $314 \pm 1$ |
| 13 | $37.5 \pm 0.1$ | $148.3 \pm 0.05$ | $13.8 \pm 0.1$ | $17.67 \pm 0.04$ | $312 \pm 1$ |
| 14 | $38.5 \pm 0.1$ | $148.0 \pm 0.05$ | $14.8 \pm 0.1$ | $17.70 \pm 0.04$ | $313 \pm 1$ |
| 15 | $39.5 \pm 0.1$ | $143.9 \pm 0.05$ | $15.8 \pm 0.1$ | $18.21 \pm 0.04$ | $332 \pm 1$ |
| 16 | $40.5 \pm 0.1$ | $141.9 \pm 0.05$ | $16.8 \pm 0.1$ | $18.46 \pm 0.04$ | $341 \pm 1$ |
| 17 | $41.5 \pm 0.1$ | $142.9 \pm 0.05$ | $17.8 \pm 0.1$ | $18.33 \pm 0.04$ | $336 \pm 1$ |
| 18 | $42.5 \pm 0.1$ | $141.9 \pm 0.05$ | $18.8 \pm 0.1$ | $18.46 \pm 0.04$ | $341 \pm 1$ |
| 19 | $43.5 \pm 0.1$ | $142.8 \pm 0.05$ | $19.8 \pm 0.1$ | $18.35 \pm 0.04$ | $337 \pm 1$ |
| 20 | $44.5 \pm 0.1$ | $144.3 \pm 0.05$ | $20.8 \pm 0.1$ | $18.16 \pm 0.04$ | $330 \pm 1$ |
| 21 | $45.5 \pm 0.1$ | $142.2 \pm 0.05$ | $21.8 \pm 0.1$ | $18.42 \pm 0.04$ | $339 \pm 1$ |
| 22 | $46.5 \pm 0.1$ | $139.8 \pm 0.05$ | $22.8 \pm 0.1$ | $18.74 \pm 0.04$ | $351 \pm 1$ |
| 23 | $47.5 \pm 0.1$ | $136.7 \pm 0.05$ | $23.8 \pm 0.1$ | $19.17 \pm 0.04$ | $368 \pm 1$ |
| 24 | $48.5 \pm 0.1$ | $133.0 \pm 0.05$ | $24.8 \pm 0.1$ | $19.70 \pm 0.04$ | $388 \pm 1$ |
| 25 | $49.5 \pm 0.1$ | $129.5 \pm 0.05$ | $25.8 \pm 0.1$ | $20.23 \pm 0.04$ | $409 \pm 1$ |
| 26 | $50.5 \pm 0.1$ | $125.7 \pm 0.05$ | $26.8 \pm 0.1$ | $20.84 \pm 0.04$ | $434 \pm 1$ |
| 27 | $51.5 \pm 0.1$ | $124.3 \pm 0.05$ | $27.8 \pm 0.1$ | $21.08 \pm 0.04$ | $444 \pm 1$ |
| 28 | $52.5 \pm 0.1$ | $123.4 \pm 0.05$ | $28.8 \pm 0.1$ | $21.23 \pm 0.04$ | $451 \pm 1$ |
| 29 | $53.5 \pm 0.1$ | $120.9 \pm 0.05$ | $29.8 \pm 0.1$ | $21.67 \pm 0.04$ | $470 \pm 1$ |
| 30 | $54.5 \pm 0.1$ | $117.5 \pm 0.05$ | $30.8 \pm 0.1$ | $22.30 \pm 0.04$ | $497 \pm 1$ |
| 31 | $55.5 \pm 0.1$ | $114.0 \pm 0.05$ | $31.8 \pm 0.1$ | $22.98 \pm 0.04$ | $528 \pm 1$ |
| 32 | $56.5 \pm 0.1$ | $111.2 \pm 0.05$ | $32.8 \pm 0.1$ | $23.56 \pm 0.05$ | $555 \pm 2$ |
| 33 | $57.5 \pm 0.1$ | $110.5 \pm 0.05$ | $33.8 \pm 0.1$ | $23.71 \pm 0.05$ | $562 \pm 2$ |
| 34 | $58.5 \pm 0.1$ | $108.1 \pm 0.05$ | $34.8 \pm 0.1$ | $24.24 \pm 0.05$ | $588 \pm 2$ |
| 35 | $59.5 \pm 0.1$ | $107.1 \pm 0.05$ | $35.8 \pm 0.1$ | $24.46 \pm 0.05$ | $598 \pm 2$ |
| 36 | $60.5 \pm 0.1$ | $104.6 \pm 0.05$ | $36.8 \pm 0.1$ | $25.05 \pm 0.05$ | $628 \pm 2$ |
| 37 | $61.5 \pm 0.1$ | $102.1 \pm 0.05$ | $37.8 \pm 0.1$ | $25.66 \pm 0.05$ | $658 \pm 2$ |
| 38 | $62.5 \pm 0.1$ | $100.1 \pm 0.05$ | $38.8 \pm 0.1$ | $26.17 \pm 0.05$ | $685 \pm 2$ |
| 39 | $63.5 \pm 0.1$ | $99.6 \pm 0.05$ | $39.8 \pm 0.1$ | $26.31 \pm 0.05$ | $692 \pm 2$ |
| 40 | $64.5 \pm 0.1$ | $97.3 \pm 0.05$ | $40.8 \pm 0.1$ | $26.93 \pm 0.05$ | $725 \pm 2$ |
| 41 | $65.5 \pm 0.1$ | $95.8 \pm 0.05$ | $41.8 \pm 0.1$ | $27.35 \pm 0.05$ | $748 \pm 2$ |
| 42 | $66.5 \pm 0.1$ | $94.7 \pm 0.05$ | $42.8 \pm 0.1$ | $27.67 \pm 0.05$ | $766 \pm 2$ |
| 43 | $67.5 \pm 0.1$ | $94.0 \pm 0.05$ | $43.8 \pm 0.1$ | $27.87 \pm 0.06$ | $777 \pm 2$ |
| 44 | $68.5 \pm 0.1$ | $92.9 \pm 0.05$ | $44.8 \pm 0.1$ | $28.20 \pm 0.06$ | $795 \pm 2$ |
| 45 | $69.5 \pm 0.1$ | $91.1 \pm 0.05$ | $45.8 \pm 0.1$ | $28.76 \pm 0.06$ | $827 \pm 2$ |

where $\quad{ }^{\text {a) }} h_{1}$ is the reading of the top position of the weight before it starts to fall,
${ }^{\text {b) }} h$ is the distance of fall of the weight which is obtained by $h=h_{1}-h_{2}+d / 2$, $h_{2}\left(=(25 \pm 0.05) \times 10^{-2} \mathrm{~m}\right)$ is the top position of the weight at the start of blocking of the photogate, $d\left(=(2.62 \pm 0.005) \times 10^{-2} \mathrm{~m}\right)$ is the length of the weight, and
${ }^{\text {c) }} v$ is obtained from $v=d / \Delta t$.
3. Relate the coefficient $C$ to the parameters of the MBB. (1.0 points)

$$
h=C v^{2}, \text { where } C=\left\{m_{0}+I / R^{2}+m\left(l^{2}+2 / 5 r^{2}\right) / R^{2}\right\} / 2 m_{0} g
$$

(Explanation) The ball is at static equilibrium $(x=l)$. When the speed of the weight is $v$, the increase in kinetic energy of the whole system is given by

$$
\begin{aligned}
\Delta K & =1 / 2 m_{\mathrm{o}} v^{2}+1 / 2 I \omega^{2}+1 / 2 m\left(l^{2}+2 / 5 r^{2}\right) \omega^{2} \\
& =1 / 2\left\{m_{0}+I / R^{2}+m\left(l^{2}+2 / 5 r^{2}\right) / R^{2}\right\} v^{2},
\end{aligned}
$$

where $\omega(=v / R)$ is the angular velocity of the Mechanical "Black Box" and $I$ is the effective moment of inertia of the whole system except the ball. Since the decrease in gravitational potential energy of the weight is

$$
\Delta U=-m_{0} g h,
$$

the energy conservation $(\Delta K+\Delta U=0)$ gives

$$
\begin{aligned}
h & =1 / 2\left\{m_{\mathrm{o}}+I / R^{2}+m\left(l^{2}+2 / 5 r^{2}\right) / \mathrm{R}^{2}\right\} v^{2} / m_{0} g \\
& =C v^{2}, \quad \text { where } C=\left\{m_{0}+I / R^{2}+m\left(l^{2}+2 / 5 r^{2}\right) / \mathrm{R}^{2}\right\} / 2 m_{0} g
\end{aligned}
$$

4. Relate the coefficients $A$ and $B$ to the parameters of the MBB. (1.0 points)

$$
\begin{aligned}
h=A v^{2}+B, \text { where } A=[ & \left.m_{0}+I / R^{2}+m\left\{(L / 2-\delta-r)^{2}+2 / 5 r^{2}\right\} / R^{2}\right] / 2 m_{0} g \\
\text { and } B=[ & -k_{1}(L / 2-l-\delta-r)^{2} \\
& \left.+k_{2}\left\{(L-2 \delta-2 r)^{2}-(L / 2+l-\delta-r)^{2}\right\}\right] / 2 m_{\circ} g
\end{aligned}
$$

(Explanation) The ball stays at the end cap of the tube $(x=L / 2-\delta-r)$. When the speed of the weight is $v$, the increase in kinetic energy of the whole system is given by

$$
K=1 / 2\left[m_{0}+I / R^{2}+m\left\{(L / 2-\delta-r)^{2}+2 / 5 r^{2}\right\} / R^{2}\right] v^{2} .
$$

Since the increase in elastic potential energy of the springs is

$$
\begin{aligned}
\Delta U_{\mathrm{e}}=1 / 2[ & -k_{1}(L / 2-l-\delta-r)^{2} \\
& \left.+k_{2}\left\{(L-2 \delta-2 r)^{2}-(L / 2+l-\delta-r)^{2}\right\}\right],
\end{aligned}
$$

the energy conservation $\left(K+\Delta U+\Delta U_{\mathrm{e}}=0\right)$ gives

$$
\begin{aligned}
& h=1 / 2\left[m_{\mathrm{o}}+I / R^{2}+m\left\{(L / 2-\delta-r)^{2}+2 / 5 r^{2}\right\} / R^{2}\right] v^{2} / m_{\mathrm{o}} g+\Delta U_{\mathrm{e}} / m_{\mathrm{o}} g \\
& =A v^{2}+B,
\end{aligned}
$$

where

$$
A=\left[m_{\mathrm{o}}+I / R^{2}+m\left\{(L / 2-\delta-r)^{2}+2 / 5 r^{2}\right\} / R^{2}\right] / 2 m_{0} g
$$

and

$$
\begin{aligned}
B= & {\left[-k_{1}(L / 2-l-\delta-r)^{2}\right.} \\
& \left.+k_{2}\left\{(L-2 \delta-2 r)^{2}-(L / 2+l-\delta-r)^{2}\right\}\right] / 2 m_{0} g .
\end{aligned}
$$

5. Determine the value of $m$ from your measurements and the results obtained in PART-A. (3.0 points)

$$
m=6.2 \times 10^{-2} \mathrm{~kg}
$$

(Explanation) From the results obtained in PART-B 3 and 4 we get

$$
A-C=\frac{m}{2 g m_{o} R^{2}}\left\{(L / 2-\delta-r)^{2}-l^{2}\right\} .
$$

The measured values are

$$
\begin{aligned}
& L=(40.0 \pm 0.05) \times 10^{-2} \mathrm{~m} \\
& m_{\mathrm{o}}=(100.4 \pm 0.05) \times 10^{-3} \mathrm{~kg} \\
& 2 R=(3.91 \pm 0.005) \times 10^{-2} \mathrm{~m}
\end{aligned}
$$

Therefore,

$$
(L / 2-\delta-r)^{2}=\{(20.0 \pm 0.03)-0.5-1.1\}^{2} \times 10^{-4} \mathrm{~m}^{2}=(338.6 \pm 0.8) \times 10^{-4} \mathrm{~m}^{2}
$$

and

$$
\begin{aligned}
2 g m_{0} R^{2} & =2 \times 980 \times(100.4 \pm 0.05) \times(1.955 \pm 0.003)^{2} \times 10^{-6} \mathrm{~kg} \cdot \mathrm{~m}^{3} / \mathrm{s}^{2} \\
& =(752 \pm 2) \times 10^{-6} \mathrm{~kg} \cdot \mathrm{~m}^{3} / \mathrm{s}^{2}
\end{aligned}
$$

The slopes of the two straight lines in the graph (Fig. B-1) of PART-B 1 are

$$
A=5.0 \pm 0.1 \mathrm{~s}^{2} / \mathrm{m} \quad \text { and } \quad C=2.4 \pm 0.1 \mathrm{~s}^{2} / \mathrm{m},
$$

respectively, and

$$
A-C=2.6 \pm 0.1 \mathrm{~s}^{2} / \mathrm{m}
$$

Since we already obtained $m \times l=(M+m) \times l_{\mathrm{cm}}=2.96 \times 10^{-3} \mathrm{~kg} \cdot \mathrm{~m}$ from PART- $\boldsymbol{A}$, the equation

$$
(338.6 \pm 0.8) m^{2}-(752 \pm 2) \times 10^{3} \times(0.026 \pm 0.001) m-(296 \pm 8)^{2}=0
$$

or

$$
(338.6 \pm 0.8) m^{2}-(19600 \pm 800) m-(88000 \pm 3000)=0
$$

is resulted, where $m$ is expressed in the unit of $g$.
The roots of this equation are

$$
m=\frac{(9800 \pm 400) \pm \sqrt{(9800 \pm 400)^{2}+(338.6 \pm 0.8) \times(88000 \pm 3000)}}{(338.6 \pm 0.8)}
$$

The physically meaningful positive root is

$$
m=\frac{(9800 \pm 400)+\sqrt{(126000000 \pm 6000000)}}{(338.6 \pm 0.8)}=(62 \pm 2) \mathrm{g}=(6.2 \pm 0.2) \times 10^{-2} \mathrm{~kg} .
$$

## PART-C The spring constants $k_{1}$ and $k_{2} \quad$ ( 6.0 points)

1. Measure the periods $T_{1}$ and $T_{2}$ of small oscillation shown in Figs. 3 (1) and (2) and write down their values, respectively. (1.0 points)

$$
T_{1}=1.1090 \mathrm{~s} \quad \text { and } \quad T_{2}=1.0193 \mathrm{~s}
$$

(Explanation)


Fig. C-1 Small oscillation experimental set up

The measured periods are

|  | $T_{1}(\mathrm{~s})$ |  | $T_{2}(\mathrm{~s})$ |
| :---: | :---: | :---: | :---: |
| 1 | $1.1085 \pm 0.00005$ | 1 | $1.0194 \pm 0.00005$ |
| 2 | $1.1092 \pm 0.00005$ | 2 | $1.0194 \pm 0.00005$ |
| 3 | $1.1089 \pm 0.00005$ | 3 | $1.0193 \pm 0.00005$ |
| 4 | $1.1085 \pm 0.00005$ | 4 | $1.0191 \pm 0.00005$ |
| 5 | $1.1094 \pm 0.00005$ | 5 | $1.0192 \pm 0.00005$ |
| 6 | $1.1090 \pm 0.00005$ | 6 | $1.0194 \pm 0.00005$ |
| 7 | $1.1088 \pm 0.00005$ | 7 | $1.0194 \pm 0.00005$ |
| 8 | $1.1090 \pm 0.00005$ | 8 | $1.0191 \pm 0.00005$ |
| 9 | $1.1092 \pm 0.00005$ | 9 | $1.0192 \pm 0.00005$ |
| 10 | $1.1094 \pm 0.00005$ | 10 | $1.0193 \pm 0.00005$ |

By averaging the 10 measurements for each configuration, respectively, we get

$$
T_{1}=1.1090 \pm 0.0003 \mathrm{~s} \quad \text { and } \quad T_{2}=1.0193 \pm 0.0001 \mathrm{~s}
$$

2. Explain, by using equations, why the angular frequencies $\omega_{1}$ and $\omega_{2}$ of small oscillation of the configurations are different. (1.0 points)

$$
\begin{gathered}
\omega_{1}=\sqrt{\frac{M g L / 2+m g(L / 2+l+\Delta l)}{I_{o}+m\left\{(L / 2+l+\Delta l)^{2}+\frac{2}{5} r^{2}\right\}}} \\
\omega_{2}=\sqrt{\frac{M g L / 2+m g(L / 2-l+\Delta l)}{I_{o}+m\left\{(L / 2-l+\Delta l)^{2}+\frac{2}{5} r^{2}\right\}}}
\end{gathered}
$$

(Explanation) The moment of inertia of the Mechanical "Black Box" with respect to the pivot at the top of the tube is

$$
I_{1}=I_{o}+m\left\{(L / 2+l+\Delta l)^{2}+\frac{2}{5} r^{2}\right\} \quad \text { or } \quad I_{2}=I_{o}+m\left\{(L / 2-l+\Delta l)^{2}+\frac{2}{5} r^{2}\right\}
$$

depending on the orientation of the MBB as shown in Figs. C-1(1) and (2), respectively.
When the MBB is slightly tilted by an angle $\theta$ from vertical, the torque applied by the gravity is

$$
\tau_{1}=M g(L / 2) \sin \theta+m g(L / 2+l+\Delta l) \sin \theta \approx\{M g(L / 2)+m g(L / 2+l+\Delta l)\} \theta
$$

or

$$
\tau_{2}=M g(L / 2) \sin \theta+m g(L / 2-l+\Delta l) \sin \theta \approx\{M g(L / 2)+m g(L / 2-l+\Delta l)\} \theta
$$

depending on the orientation.
Therefore, the angular frequencies of oscillation become

$$
\omega_{1}=\sqrt{\frac{\tau_{1} / \theta}{I_{1}}}=\sqrt{\frac{M g L / 2+m g(L / 2+l+\Delta l)}{I_{o}+m\left\{(L / 2+l+\Delta l)^{2}+\frac{2}{5} r^{2}\right\}}}
$$

and

$$
\omega_{2}=\sqrt{\frac{\tau_{2} / \theta}{I_{2}}}=\sqrt{\frac{M g L / 2+m g(L / 2-l+\Delta l)}{I_{o}+m\left\{(L / 2-l+\Delta l)^{2}+\frac{2}{5} r^{2}\right\}}} .
$$

3. Evaluate $\Delta l$ by eliminating $I_{\mathrm{o}}$ from the previous results. (1.0 points)

$$
\Delta l=(7.2 \pm 0.9) \mathrm{cm}=(7.2 \pm 0.9) \times 10^{-2} \mathrm{~m}
$$

(Explanation) By rewriting the two expressions for the angular frequencies $\omega_{1}$ and $\omega_{2}$ as

$$
M g L / 2+m g(L / 2+l+\Delta l)=I_{o} \omega_{1}^{2}+m \omega_{1}^{2}\left\{(L / 2+l+\Delta l)^{2}+\frac{2}{5} r^{2}\right\}
$$

and

$$
M g L / 2+m g(L / 2-l+\Delta l)=I_{o} \omega_{2}^{2}+m \omega_{2}^{2}\left\{(L / 2-l+\Delta l)^{2}+\frac{2}{5} r^{2}\right\}
$$

one can eliminate the unknown moment of inertia $I_{\mathrm{o}}$ of the MBB without the ball. By eliminating the $I_{\mathrm{o}}$ one gets the equation for $\Delta l$

$$
\left(\omega_{2}^{2}-\omega_{1}^{2}\right)\left\{\frac{(M+m) g L}{2}+m g \Delta l\right\}+\left(\omega_{1}^{2}+\omega_{2}^{2}\right) m g l=\omega_{1}^{2} \omega_{2}^{2} m(L+2 \Delta l)(2 l) .
$$

From the measured or given values we get,

$$
\left.\begin{array}{rl}
\left(\omega_{2}^{2}-\omega_{1}^{2}\right) & =\left\{\left(\frac{2 \pi}{T_{2}}\right)^{2}-\left(\frac{2 \pi}{T_{1}}\right)^{2}\right\}=\left(\frac{6.2832}{1.0193 \pm 0.0001}\right)^{2}-\left(\frac{6.2832}{1.1090 \pm 0.0003}\right)^{2} \\
= & 5.90 \pm 0.01 \mathrm{~s}^{-2}
\end{array}\right) \begin{aligned}
\frac{(M+m) g L}{2} & =\frac{(141.1 \pm 0.05) \times 980 \times(40.0 \pm 0.05)}{2}=(27.66 \pm 0.04) \times 10^{-2} \mathrm{~kg} \cdot \mathrm{~m}^{2} / \mathrm{s}^{2} \\
\left(\omega_{1}^{2}+\omega_{2}^{2}\right) m g l & =\left\{\left(\frac{2 \pi}{T_{1}}\right)^{2}+\left(\frac{2 \pi}{T_{2}}\right)^{2}\right\}(M+m) l_{c m} g \\
& =\left\{\left(\frac{6.2832}{1.1090 \pm 0.0003}\right)^{2}+\left(\frac{6.2832}{1.0193 \pm 0.0001}\right)^{2}\right\} \times(296 \pm 8) \times 980
\end{aligned}
$$

$$
=(203 \pm 5) \times 10^{-2} \mathrm{~kg} \cdot \mathrm{~m}^{2} / \mathrm{s}^{4}
$$

$$
\begin{aligned}
\omega_{1}^{2} \omega_{2}^{2} m l & =\left(\frac{2 \pi}{T_{1}}\right)^{2}\left(\frac{2 \pi}{T_{2}}\right)^{2}(M+m) l_{c m} \\
& =\left(\frac{6.2832}{1.1090 \pm 0.0003}\right)^{2}\left(\frac{6.2832}{1.0193 \pm 0.0001}\right)^{2} \times(296 \pm 8) \\
& =(3.6 \pm 0.1) \mathrm{kg} \cdot \mathrm{~m} / \mathrm{s}^{4}
\end{aligned}
$$

Therefore, the equation we obtained in PART-C 3 becomes

$$
\begin{aligned}
&(5.90 \pm 0.01)\left\{(27.66 \pm 0.04) \times 10^{5}+(62 \pm 2) \times 980 \times \Delta l\right\}+(203 \pm 5) \times 10^{5} \\
&=(7.2 \pm 0.2) \times 10^{5} \times\{(40.0 \pm 0.05)+2 \Delta l\}
\end{aligned}
$$

where $\Delta l$ is expressed in the unit of cm . By solving the equation we get

$$
\Delta l=(7.2 \pm 0.9) \mathrm{cm}=(7.2 \pm 0.9) \times 10^{-2} \mathrm{~m}
$$

4. Write down the value of the effective total spring constant $k$ of the two-spring system. (2.0 points)

$$
k=9 \mathrm{~N} / \mathrm{m}
$$

(Explanation) The effective total spring constant is

$$
k \equiv \frac{m g}{\Delta l}=\frac{(62 \pm 2) \times 980}{7.2 \pm 0.9}=9000 \pm 1000 \text { dyne } / \mathrm{cm} \quad \text { or } \quad 9 \pm 1 \mathrm{~N} / \mathrm{m} .
$$

5. Obtain the respective values of $k_{1}$ and $k_{2}$. Write down their values. (1.0 point)

$$
\begin{aligned}
& k_{1}=5.7 \mathrm{~N} / \mathrm{m} \\
& k_{2}=3 \mathrm{~N} / \mathrm{m}
\end{aligned}
$$

(Explanation) When the MBB is in equilibrium on a horizontal plane the force balance condition for the ball is that

$$
\frac{L / 2-l-\delta-r}{L / 2+l-\delta-r}=\frac{N_{1}}{N_{2}}=\frac{k_{2}}{k_{1}} .
$$

Since $k=k_{1}+k_{2}$, we get

$$
k_{1}=\frac{k}{\frac{L / 2-l-\delta-r}{L / 2+l-\delta-r}+1}=\frac{L / 2+l-\delta-r}{L-2 \delta-2 r} k
$$

and

$$
k_{2}=k-k_{1}=\frac{L / 2-l-\delta-r}{L-2 \delta-2 r} k .
$$

From the measured or given values

$$
\frac{L / 2+l-\delta-r}{L-2 \delta-2 r}=\frac{(20.0 \pm 0.03)+\left(\frac{296 \pm 8}{62 \pm 2}\right)-0.5-1.1}{(40.0 \pm 0.05)-1.0-2.2}=0.63 \pm 0.005
$$

Therefore,

$$
k_{1}=(0.63 \pm 0.005) \times(9000 \pm 1000)=5700 \pm 600 \text { dyne } / \mathrm{cm} \quad \text { or } \quad 5.7 \pm 0.6 \mathrm{~N} / \mathrm{m},
$$

and

$$
k_{2}=(9000 \pm 1000)-(5700 \pm 600)=3000 \pm 1000 \text { dyne } / \mathrm{cm} \quad \text { or } \quad 3 \pm 1 \mathrm{~N} / \mathrm{m} .
$$


[^0]:    ${ }^{\S} 2 B_{0}$ is a material property called remanent magnetic induction, $B_{r}$.

[^1]:    ${ }^{1}$ Authors: Leó Kristjánsson and Thorsteinn Vilhjálmsson

[^2]:    ${ }^{2}$ This may also be done by using Steiner's theorem twice, going from the previous axis of impact to the center of mass and from there to the new axis of impact.
    ${ }^{3}$ Alternatively:

[^3]:    ${ }^{4}$ In the general case $\alpha=2 \pi / N$.

[^4]:    ${ }^{5}$ You can of course solve any of the inequalities in a purely numerical way, e.g. by progressive guessing or by using the approximations $\sin \phi \approx \phi$ and $\cos \phi \approx 1-\phi^{2} / 2$.

[^5]:    ${ }^{6}$ Authors: Gudni Axelsson and Thorsteinn Vilhjálmsson

[^6]:    ${ }^{7}$ Authors: Einar Gudmundsson, Knútur Árnason and Thorsteinn Vilhjálmsson

[^7]:    These problems have been prepared by the Scientific Committee of the 30th IPhO, including professors at the Universities of Bologna, Naples, Turin and Trieste.

[^8]:    ${ }^{1}$ Throughout this problem $\alpha$ is expressed in radians
    Final

[^9]:    This problem has been prepared by the Scientific Committee of the 30th IPhO, including professors at the Universities of Bologna, Naples, Turin and Trieste.

[^10]:    ${ }^{1}$ Including the small hex locking nut.

[^11]:    ${ }^{2}$ The small hex nut must be locked in place every time you move the threaded rod. Its mass is included in $M_{1}$. This locking must be repeated also in the following, each time you move the threaded rod.

[^12]:    ${ }^{3}$ In order to stabilize it in this position, you may reposition the stand brackets.

[^13]:    ${ }^{4}$ You may be able to observe two equilibrium positions, but one of them is more stable than the other (see figure 5). Report and plot the period for the more stable one.

[^14]:    ${ }^{1}$ Mr. Maurizio Recchi.
    ${ }^{2}$ This can easily be done by balancing the pendulum, e.g. on the T-shaped rod provided.

