OMEGALEPH INSTITUTE FOR ADVANCED EDUCATION

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International Mathematical Olympiads 1959-2011

I-XLIII - IMO 1959-2011

OMEGALEPH

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International Mathematical Olympiads 1959-2011

IMO 1959-2011

Omegaleph Compilations

IMO 1959

Brasov and Bucharest, Romania

Day 1

$\boxed{1}$ Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number n.

$\boxed{2}$ For what real values of x is

$$\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=A$$

given

a) $A=\sqrt{2}$;

b) $A=1$;

c) $A=2$,

where only non-negative real numbers are admitted for square roots?

$\boxed{3}$ Let a, b, c be real numbers. Consider the quadratic equation in $\cos x$

$$a\cos x^2+b\cos x+c=0.$$

Using the numbers a, b, c form a quadratic equation in $\cos 2x$ whose roots are the same as those of the original equation. Compare the equation in $\cos x$ and $\cos 2x$ for $a=4$, $b=2$, $c=-1$.

IMO 1959

Brasov and Bucharest, Romania

Day 2

$\boxed{4}$ Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.

$\boxed{5}$ An arbitrary point M is selected in the interior of the segment AB. The square $AMCD$ and $MBEF$ are constructed on the same side of AB, with segments AM and MB as their respective bases. The circles circumscribed about these squares, with centers P and Q, intersect at M and also at another point N. Let N' denote the point of intersection of the straight lines AF and BC.

a) Prove that N and N' coincide;

b) Prove that the straight lines MN pass through a fixed point S independent of the choice of M;

c) Find the locus of the midpoints of the segments PQ as M varies between A and B.

$\boxed{6}$ Two planes, P and Q, intersect along the line p. The point A is given in the plane P, and the point C in the plane Q; neither of these points lies on the straight line p. Construct an isosceles trapezoid $ABCD$ (with $AB \parallel CD$) in which a circle can be inscribed, and with vertices B and D lying in planes P and Q respectively.

IMO 1960

Sinaia, Romania

Day 1

$\boxed{1}$ Determine all three-digit numbers N having the property that N is divisible by 11, and $\dfrac{N}{11}$ is equal to the sum of the squares of the digits of N.

$\boxed{2}$ For what values of the variable x does the following inequality hold:

$$\frac{4x^2}{(1-\sqrt{2x+1})^2} < 2x+9 \ ?$$

$\boxed{3}$ In a given right triangle ABC, the hypotenuse BC, of length a, is divided into n equal parts (n and odd integer). Let α be the acute angel subtending, from A, that segment which contains the mdipoint of the hypotenuse. Let h be the length of the altitude to the hypotenuse fo the triangle. Prove that:

$$\tan\alpha = \frac{4nh}{(n^2-1)a}.$$

IMO 1960

Sinaia, Romania

Day 2

$\boxed{4}$ Construct triangle ABC, given h_a, h_b (the altitudes from A and B), and m_a, the median from vertex A.

$\boxed{5}$ Consider the cube $ABCDA'B'C'D'$ (with face $ABCD$ directly above face $A'B'C'D'$).

a) Find the locus of the midpoints of the segments XY, where X is any point of AC and Y is any piont of $B'D'$;

b) Find the locus of points Z which lie on the segment XY of part a) with $ZY = 2XZ$.

$\boxed{6}$ Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. let V_1 be the volume of the cone and V_2 be the volume of the cylinder.

a) Prove that $V_1 \neq V_2$;

b) Find the smallest number k for which $V_1 = kV_2$; for this case, construct the angle subtended by a diamter of the base of the cone at the vertex of the cone.

$\boxed{7}$ An isosceles trapezoid with bases a and c and altitude h is given.

a) On the axis of symmetry of this trapezoid, find all points P such that both legs of the trapezoid subtend right angles at P;

b) Calculate the distance of p from either base;

c) Determine under what conditions such points P actually exist. Discuss various cases that might arise.

IMO 1961

Veszprem, Hungary

Day 1

1. Solve the system of equations:

$$x + y + z = a$$
$$x^2 + y^2 + z^2 = b^2$$
$$xy = z^2$$

where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z are distinct positive numbers.

2. Let a, b, c be the sides of a triangle, and S its area. Prove:

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3}$$

In what case does equality hold?

3. Solve the equation $\cos^n x - \sin^n x = 1$ where n is a natural number.

IMO 1961

Veszprem, Hungary

Day 2

4 Consider triangle $P_1P_2P_3$ and a point p within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

at least one is ≤ 2 and at least one is ≥ 2

5 Construct a triangle ABC if $AC = b$, $AB = c$ and $\angle AMB = w$, where M is the midpoint of the segment BC and $w < 90$. Prove that a solution exists if and only if

$$b \tan \frac{w}{2} \leq c < b$$

In what case does the equality hold?

6 Consider a plane ϵ and three non-collinear points A, B, C on the same side of ϵ; suppose the plane determined by these three points is not parallel to ϵ. In plane ϵ take three arbitrary points A', B', C'. Let L, M, N be the midpoints of segments AA', BB', CC'; Let G be the centroid of the triangle LMN. (We will not consider positions of the points A', B', C' such that the points L, M, N do not form a triangle.) What is the locus of point G as A', B', C' range independently over the plane ϵ?

IMO 1962

Ceske Budejovice, Czechoslovakia

Day 1

1. Find the smallest natural number n which has the following properties:

   a) Its decimal representation has a 6 as the last digit.

   b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

2. Determine all real numbers x which satisfy the inequality:

$$\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}$$

3. Consider the cube $ABCDA'B'C'D'$ ($ABCD$ and $A'B'C'D'$ are the upper and lower bases, repsectively, and edges AA', BB', CC', DD' are parallel). The point X moves at a constant speed along the perimeter of the square $ABCD$ in the direction $ABCDA$, and the point Y moves at the same rate along the perimiter of the square $B'C'CB$ in the direction $B'C'CBB'$. Points X and Y begin their motion at the same instant from the starting positions A and B', respectively. Determine and draw the locus of the midpionts of the segments XY.

IMO 1962

Ceske Budejovice, Czechoslovakia

Day 2

$\boxed{4}$ Solve the equation $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$

$\boxed{5}$ On the circle K there are given three distinct points A, B, C. Construct (using only a straightedge and a compass) a fourth point D on K such that a circle can be inscribed in the quadrilateral thus obtained.

$\boxed{6}$ Consider an isosceles triangle. let R be the radius of its circumscribed circle and r be the radius of its inscribed circle. Prove that the distance d between the centers of these two circle is

$$d = \sqrt{R(R - 2r)}$$

$\boxed{7}$ The tetrahedron $SABC$ has the following property: there exist five spheres, each tangent to the edges SA, SB, SC, BC, CA, AB, or to their extensions.

a) Prove that the tetrahedron $SABC$ is regular.

b) Prove conversely that for every regular tetrahedron five such spheres exist.

IMO 1963

Warsaw, Poland

Day 1

$\boxed{1}$ Find all real roots of the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$$

where p is a real parameter.

$\boxed{2}$ Point A and segment BC are given. Determine the locus of points in space which are vertices of right angles with one side passing through A, and the other side intersecting segment BC.

$\boxed{3}$ In an n-gon $A_1A_2\dots A_n$, all of whose interior angles are equal, the lengths of consecutive sides satisfy the relation

$$a_1 \geq a_2 \geq \dots \geq a_n.$$

Prove that $a_1 = a_2 = \ldots = a_n$.

Day 2

4 Find all solutions x_1, x_2, x_3, x_4, x_5 of the system

$$x_5 + x_2 = yx_1$$

$$x_1 + x_3 = yx_2$$

$$x_2 + x_4 = yx_3$$

$$x_3 + x_5 = yx_4$$

$$x_4 + x_1 = yx_5$$

where y is a parameter.

5 Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$

6 Five students A, B, C, D, E took part in a contest. One prediction was that the contestants would finish in the order $ABCDE$. This prediction was very poor. In fact, no contestant finished in the position predicted, and no two contestants predicted to finish consecutively actually did so. A second prediction had the contestants finishing in the order $DAECB$. This prediction was better. Exactly two of the contestants finished in the places predicted, and two disjoint pairs of students predicted to finish consecutively actually did so. Determine the order in which the contestants finished.

IMO 1964

Moscow, USSR

Day 1

$\boxed{2}$ Suppose a, b, c are the sides of a triangle. Prove that

$$a^2(b+c-a)+b^2(a+c-b)+c^2(a+b-c)\leq 3abc$$

$\boxed{3}$ A circle is inscribed in a triangle ABC with sides a, b, c. Tangents to the circle parallel to the sides of the triangle are contructe. Each of these tangents cuts off a triagnle from $\triangle ABC$. In each of these triangles, a circle is inscribed. Find the sum of the areas of all four inscribed circles (in terms of a, b, c).

Day 2

4 Seventeen people correspond by mail with one another-each one with all the rest. In their letters only three different topics are discussed. each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

5 Supppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maxium number of intersections that these perpendiculars can have.

6 In tetrahedron $ABCD$, vertex D is connected with D_0, the centrod if $\triangle ABC$. Line parallel to DD_0 are drawn through A, B and C. These lines intersect the planes BCD, CAD and ABD in points A_2, B_1, and C_1, respectively. Prove that the volume of $ABCD$ is one third the volume of $A_1B_1C_1D_0$. Is the result if point D_o is selected anywhere within $\triangle ABC$?

IMO 1965

Berlin, German Democratic Republic

Day 1

$\boxed{1}$ Determine all values of x in the interval $0 \leq x \leq 2\pi$ which satisfy the inequality

$$2\cos x \leq \sqrt{1+\sin 2x} - \sqrt{1-\sin 2x} \leq \sqrt{2}.$$

$\boxed{2}$ Consider the sytem of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

with unknowns x_1, x_2, x_3. The coefficients satisfy the conditions:

a) a_{11}, a_{22}, a_{33} are positive numbers;

b) the remaining coefficients are negative numbers;

c) in each equation, the sum ofthe coefficients is positive.

Prove that the given system has only the solution $x_1 = x_2 = x_3 = 0$.

$\boxed{3}$ Given the tetrahedron $ABCD$ whose edges AB and CD have lengths a and b respectively. The distance between the skew lines AB and CD is d, and the angle between them is ω. Tetrahedron $ABCD$ is divided into two solids by plane ϵ, parallel to lines AB and CD. The ratio of the distances of ϵ from AB and CD is equal to k. Compute the ratio of the volumes of the two solids obtained.

IMO 1965

Berlin, German Democratic Republic

Day 2

$\boxed{4}$ Find all sets of four real numbers x_1, x_2, x_3, x_4 such that the sum of any one and the product of the other three is equal to 2.

$\boxed{5}$ Consider $\triangle OAB$ with acute angle AOB. Thorugh a point $M \neq O$ perpendiculars are drawn to OA and OB, the feet of which are P and Q respectively. The point of intersection of the altitudes of $\triangle OPQ$ is H. What is the locus of H if M is permitted to range over

a) the side AB;

b) the interior of $\triangle OAB$.

$\boxed{6}$ In a plane a set of n points ($n \geq 3$) is give. Each pair of points is connected by a segment. Let d be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length d. Prove that the number of diameters of the given set is at most n.

IMO 1966

Bucharest, Romania

Day 1

$\boxed{1}$ In a mathematical contest, three problems, A, B, C were posed. Among the participants ther were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A, the number who solved B was twice the number who solved C. The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A. How many students solved only problem B?

$\boxed{2}$ Let a, b, c be the lengths of the sides of a triangle, and α, β, γ respectively, the angles opposite these sides. Prove that if

$$a + b = \tan \frac{\gamma}{2}(a \tan \alpha + b \tan \beta)$$

the triangle is isosceles.

$\boxed{3}$ Prove that the sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

Day 2

$\boxed{4}$ Prove that for every natural number n, and for every real number $x \neq \frac{k\pi}{2^t}$ $(t = 0, 1, \dots, n;\ k$ any integer)

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x$$

$\boxed{5}$ Solve the system of equations

$$|a_1 - a_2|x_2 + |a_1 - a_3|x_3 + |a_1 - a_4|x_4 = 1$$

$$|a_2 - a_1|x_1 + |a_2 - a_3|x_3 + |a_2 - a_4|x_4 = 1$$

$$|a_3 - a_1|x_1 + |a_3 - a_2|x_2 + |a_3 - a_4|x_4 = 1$$

$$|a_4 - a_1|x_1 + |a_4 - a_2|x_2 + |a_4 - a_3|x_3 = 1$$

where a_1, a_2, a_3, a_4 are four different real numbers.

$\boxed{6}$ Let ABC be a triangle, and let P, Q, R be three points in the interiors of the sides BC, CA, AB of this triangle. Prove that the area of at least one of the three triangles AQR, BRP, CPQ is less than or equal to one quarter of the area of triangle ABC.

Alternative formulation: Let ABC be a triangle, and let P, Q, R be three points on the segments BC, CA, AB, respectively. Prove that

$\min\{|AQR|, |BRP|, |CPQ|\} \leq \frac{1}{4} \cdot |ABC|$,

where the abbreviation $|P_1P_2P_3|$ denotes the (non-directed) area of an arbitrary triangle $P_1P_2P_3$.

IMO 1967

Cetinje, Yugoslavia

Day 1

$\boxed{1}$ The parallelogram $ABCD$ has $AB = a, AD = 1$, $\angle BAD = A$, and the triangle ABD has all angles acute. Prove that circles radius 1 and center A, B, C, D cover the parallelogram if and only

$$a \leq \cos A + \sqrt{3} \sin A.$$

$\boxed{2}$ Prove that a tetrahedron with just one edge length greater than 1 has volume at most $\frac{1}{8}$.

$\boxed{3}$ Let k, m, n be natural numbers such that $m + k + 1$ is a prime greater than $n + 1$. Let $c_s = s(s + 1)$. Prove that

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \ldots (c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 \ldots c_n$.

IMO 1967

Cetinje, Yugoslavia

Day 2

4 $A_0B_0C_0$ and $A_1B_1C_1$ are acute-angled triangles. Describe, and prove, how to construct the triangle ABC with the largest possible area which is circumscribed about $A_0B_0C_0$ (so BC contains B_0, CA contains B_0, and AB contains C_0) and similar to $A_1B_1C_1$.

5 Let $a_1, \ldots, a_8$ be reals, not all equal to zero. Let

$$c_n = \sum_{k=1}^{8} a_k^n$$

for $n = 1, 2, 3, \ldots$. Given that among the numbers of the sequence (c_n), there are infinitely many equal to zero, determine all the values of n for which $c_n = 0$.

6 In a sports meeting a total of m medals were awarded over n days. On the first day one medal and $\frac{1}{7}$ of the remaining medals were awarded. On the second day two medals and $\frac{1}{7}$ of the remaining medals were awarded, and so on. On the last day, the remaining n medals were awarded. How many medals did the meeting last, and what was the total number of medals ?

IMO 1968

Moscow, USSR

Day 1

1. Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.

2. Find all natural numbers n the product of whose decimal digits is $n^2 - 10n - 22$.

3. Let a, b, c be real numbers with a non-zero. It is known that the real numbers $x_1, x_2, \dots, x_n$ satisfy the n equations:

$$\begin{aligned} ax_1^2 + bx_1 + c &= x_2 \\ ax_2^2 + bx_2 + c &= x_3 \\ \dots \quad \dots \quad & \dots \quad \dots \\ ax_n^2 + bx_n + c &= x_1 \end{aligned}$$

Prove that the system has **zero**, one or *more than one* real solutions if $(b-1)^2 - 4ac$ is **negative**, equal to zero or *positive* respectively.

IMO 1968

Moscow, USSR

Day 2

$\boxed{4}$ Prove that every tetrahedron has a vertex whose three edges have the right lengths to form a triangle.

$\boxed{5}$ Let f be a real-valued function defined for all real numbers, such that for some $a > 0$ we have

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}$$

for all x. Prove that f is periodic, and give an example of such a non-constant f for $a = 1$.

$\boxed{6}$ Let n be a natural number. Prove that

$$\left\lfloor \frac{n+2^0}{2^1} \right\rfloor + \left\lfloor \frac{n+2^1}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n+2^{n-1}}{2^n} \right\rfloor = n.$$

[hide="Remark"]For any real number x, the number $\lfloor x \rfloor$ represents the largest integer smaller or equal with x.

IMO 1969

Bucharest, Romania

Day 1

1. Prove that there are infinitely many positive integers m, such that $n^4 + m$ is not prime for any positive integer n.

2. Let $f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) + \ldots + \frac{1}{2^{n-1}}\cos(a_n + x)$, where a_i are real constants and x is a real variable. If $f(x_1) = f(x_2) = 0$, prove that $x_1 - x_2$ is a multiple of π.

3. For each of $k = 1, 2, 3, 4, 5$ find necessary and sufficient conditions on $a > 0$ such that there exists a tetrahedron with k edges length a and the remainder length 1.

IMO 1969

Bucharest, Romania

Day 2

4. C is a point on the semicircle diameter AB, between A and B. D is the foot of the perpendicular from C to AB. The circle K_1 is the incircle of ABC, the circle K_2 touches CD, DA and the semicircle, the circle K_3 touches CD, DB and the semicircle. Prove that K_1, K_2 and K_3 have another common tangent apart from AB.

5. Given $n > 4$ points in the plane, no three collinear. Prove that there are at least $\frac{(n-3)(n-4)}{2}$ convex quadrilaterals with vertices amongst the n points.

6. Given real numbers $x_1, x_2, y_1, y_2, z_1, z_2$ satisfying $x_1 > 0, x_2 > 0, x_1y_1 > z_1^2$, and $x_2y_2 > z_2^2$, prove that:
$$\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \leq \frac{1}{x_1y_1-z_1^2} + \frac{1}{x_2y_2-z_2^2}.$$
Give necessary and sufficient conditions for equality.

IMO 1970

Day 1

$\boxed{1}$ M is any point on the side AB of the triangle ABC. r, r_1, r_2 are the radii of the circles inscribed in ABC, AMC, BMC. q is the radius of the circle on the opposite side of AB to C, touching the three sides of AB and the extensions of CA and CB. Similarly, q_1 and q_2. Prove that $r_1 r_2 q = r q_1 q_2$.

$\boxed{2}$ We have $0 \leq x_i < b$ for $i = 0, 1, \ldots, n$ and $x_n > 0, x_{n-1} > 0$. If $a > b$, and $x_n x_{n-1} \ldots x_0$ represents the number A base a and B base b, whilst $x_{n-1} x_{n-2} \ldots x_0$ represents the number A' base a and B' base b, prove that $A'B < AB'$.

$\boxed{3}$ The real numbers $a_0, a_1, a_2, \ldots$ satisfy $1 = a_0 \leq a_1 \leq a_2 \leq \ldots . b_1, b_2, b_3, \ldots$ are defined by $b_n = \sum_{k=1}^{n} \frac{1 - \frac{a_{k-1}}{a_k}}{\sqrt{a_k}}$.

a.) Prove that $0 \leq b_n < 2$.

b.) Given c satisfying $0 \leq c < 2$, prove that we can find a_n so that $b_n > c$ for all sufficiently large n.

IMO 1970

Day 2

$\boxed{1}$ Find all positive integers n such that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two subsets so that the product of the numbers in each subset is equal.

$\boxed{2}$ In the tetrahedron $ABCD, \angle BDC = 90^o$ and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

When do we have equality?

$\boxed{3}$ Given 100 coplanar points, no three collinear, prove that at most 70% of the triangles formed by the points have all angles acute.

IMO 1971

Day 1

$\boxed{1}$ Let

$$E_n = (a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)+(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)+\ldots+(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1}).$$

Let S_n be the proposition that $E_n \geq 0$ for all real a_i. Prove that S_n is true for $n = 3$ and 5, but for no other $n > 2$.

$\boxed{2}$ Let P_1 be a convex polyhedron with vertices $A_1, A_2, \ldots, A_9$. Let P_i be the polyhedron obtained from P_1 by a translation that moves A_1 to A_i. Prove that at least two of the polyhedra $P_1, P_2, \ldots, P_9$ have an interior point in common.

$\boxed{3}$ Prove that we can find an infinite set of positive integers of the from $2^n - 3$ (where n is a positive integer) every pair of which are relatively prime.

IMO 1971

Day 2

$\boxed{1}$ All faces of the tetrahedron $ABCD$ are acute-angled. Take a point X in the interior of the segment AB, and similarly Y in BC, Z in CD and T in AD.

a.) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then prove none of the closed paths $XYZTX$ has minimal length;

b.) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest paths $XYZTX$, each with length $2AC \sin k$, where $2k = \angle BAC + \angle CAD + \angle DAB$.

$\boxed{2}$ Prove that for every positive integer m we can find a finite set S of points in the plane, such that given any point A of S, there are exactly m points in S at unit distance from A.

$\boxed{3}$ Let $A = (a_{ij})$, where $i, j = 1, 2, \ldots, n$, be a square matrix with all a_{ij} non-negative integers. For each i, j such that $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is at least n. Prove that the sum of all the elements in the matrix is at least $\frac{n^2}{2}$.

IMO 1972

Day 1

$\boxed{1}$ Prove that from a set of ten distinct two-digit numbers, it is always possible to find two disjoint subsets whose members have the same sum.

$\boxed{2}$ Given $n > 4$, prove that every cyclic quadrilateral can be dissected into n cyclic quadrilaterals.

$\boxed{3}$ Prove that $(2m)!(2n)!$ is a multiple of $m!n!(m+n)!$ for any non-negative integers m and n.

IMO 1972

Day 2

$\boxed{1}$ Find all positive real solutions to:

$$\begin{array}{rcl} (x_1^2 - x_3x_5)(x_2^2 - x_3x_5) & \leq & 0(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) \leq \\ 0(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) & \leq & 0(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) \leq \\ 0(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) & \leq & 0 \end{array}$$

(0)

f and g are real-valued functions defined on the real line. For all x and $y, f(x+y) + f(x-y) = 2f(x)g(y)$. f is not identically zero and $|f(x)| \leq 1$ for all x. Prove that $|g(x)| \leq 1$ for all x.

Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

IMO 1973

Day 1

1. Prove that the sum of an odd number of vectors of length 1, of common origin O and all situated in the same semi-plane determined by a straight line which goes through O, is at least 1.

2. Establish if there exists a finite set M of points in space, not all situated in the same plane, so that for any straight line d which contains at least two points from M there exists another straight line d', parallel with d, but distinct from d, which also contains at least two points from M.

3. Determine the minimum value of $a^2 + b^2$ when (a, b) traverses all the pairs of real numbers for which the equation
$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$
has at least one real root.

IMO 1973

Day 2

1. A soldier needs to check if there are any mines in the interior or on the sides of an equilateral triangle ABC. His detector can detect a mine at a maximum distance equal to half the height of the triangle. The soldier leaves from one of the vertices of the triangle. Which is the minimum distance that he needs to traverse so that at the end of it he is sure that he completed successfully his mission?

2. G is a set of non-constant functions f. Each f is defined on the real line and has the form $f(x) = ax + b$ for some real a, b. If f and g are in G, then so is fg, where fg is defined by $fg(x) = f(g(x))$. If f is in G, then so is the inverse f^{-1}. If $f(x) = ax+b$, then $f^{-1}(x) = \frac{x-b}{a}$. Every f in G has a fixed point (in other words we can find x_f such that $f(x_f) = x_f$. Prove that all the functions in G have a common fixed point.

3. Let $a_1, \ldots, a_n$ be n positive numbers and $0 < q < 1$. Determine n positive numbers $b_1, \ldots, b_n$ so that:

   a.) $k < b_k$ for all $k = 1, \ldots, n$, *b.)* $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for all $k = 1, \ldots, n-1$, *c.)* $\sum\limits_{k=1}^{n} b_k < \frac{1+q}{1-q} \cdot \sum\limits_{k=1}^{n} a_k$.

IMO 1974

Erfurt and East Berlin, German Democratic Republic

Day 1

$\boxed{1}$ Three players A, B and C play a game with three cards and on each of these 3 cards it is written a positive integer, all 3 numbers are different. A game consists of shuffling the cards, giving each player a card and each player is attributed a number of points equal to the number written on the card and then they give the cards back. After a number (≥ 2) of games we find out that A has 20 points, B has 10 points and C has 9 points. We also know that in the last game B had the card with the biggest number. Who had in the first game the card with the second value (this means the middle card concerning its value).

$\boxed{2}$ Let ABC be a triangle. Prove that there exists a point D on the side AB of the triangle ABC, such that CD is the geometric mean of AD and DB, iff the triangle ABC satisfies the inequality $\sin A \sin B \leq \sin^2 \frac{C}{2}$.

[hide="Comment"]*Alternative formulation, from IMO ShortList 1974, Finland 2:* We consider a triangle ABC. Prove that: $\sin(A)\sin(B) \leq \sin^2\left(\frac{C}{2}\right)$ is a necessary and sufficient condition for the existence of a point D on the segment AB so that CD is the geometrical mean of AD and BD.

$\boxed{3}$ Prove that for any n natural, the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

cannot be divided by 5.

IMO 1974

Erfurt and East Berlin, German Democratic Republic

Day 2

$\boxed{4}$ We consider the division of a chess board 8×8 in p disjoint rectangles which satisfy the conditions:

a) every rectangle is formed from a number of full squares (not partial) from the 64 and the number of white squares is equal to the number of black squares.

b) the numbers $a_1, \ldots, a_p$ of white squares from p rectangles satisfy $a_1, , \ldots, a_p$. Find the greatest value of p for which there exists such a division and then for that value of p, all the sequences $a_1, \ldots, a_p$ for which we can have such a division.

$\boxed{5}$ The variables a, b, c, d, traverse, independently from each other, the set of positive real values. What are the values which the expression

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

takes?

$\boxed{6}$ Let $P(x)$ be a polynomial with integer coefficients. We denote $\deg(P)$ its degree which is ≥ 1. Let $n(P)$ be the number of all the integers k for which we have $(P(k))^2 = 1$. Prove that $n(P) - \deg(P) \leq 2$.

IMO 1975

Day 1

1. We consider two sequences of real numbers $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$. Let $z_1, z_2, \ldots, z_n$ be a permutation of the numbers $y_1, y_2, \ldots, y_n$. Prove that $\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$.

2. Let $a_1, \ldots, a_n$ be an infinite sequence of strictly positive integers, so that $a_k < a_{k+1}$ for any k. Prove that there exists an infinity of terms m, which can be written like $a_m = x \cdot a_p + y \cdot a_q$ with x, y strictly positive integers and $p \neq q$.

3. In the plane of a triangle ABC, in its exterior, we draw the triangles ABR, BCP, CAQ so that $\angle PBC = \angle CAQ = 45^\circ$, $\angle BCP = \angle QCA = 30^\circ$, $\angle ABR = \angle RAB = 15^\circ$.

   Prove that

   a.) $\angle QRP = 90^\circ$, and

   b.) $QR = RP$.

IMO 1975

Day 2

[4] When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

[5] Can there be drawn on a circle of radius 1 a number of 1975 distinct points, so that the distance (measured on the chord) between any two points (from the considered points) is a rational number?

[6] Determine the polynomials P of two variables so that:

a.) for any real numbers t, x, y we have $P(tx, ty) = t^n P(x, y)$ where n is a positive integer, the same for all t, x, y;

b.) for any real numbers a, b, c we have $P(a + b, c) + P(b + c, a) + P(c + a, b) = 0$;

c.) $P(1, 0) = 1$.

IMO 1976

Day 1

$\boxed{1}$ In a convex quadrilateral (in the plane) with the area of 32 cm^2 the sum of two opposite sides and a diagonal is 16 cm. Determine all the possible values that the other diagonal can have.

$\boxed{2}$ Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for j= $2, \ldots$ Prove that for any positive integer n the roots of the equation $P_n(x) = x$ are all real and distinct.

$\boxed{3}$ A box whose shape is a parallelepiped can be completely filled with cubes of side 1. If we put in it the maximum possible number of cubes, each ofvolume, 2, with the sides parallel to those of the box, then exactly 40 percent from the volume of the box is occupied. Determine the possible dimensions of the box.

IMO 1976

Day 2

$\boxed{1}$ Determine the greatest number, who is the product of some positive integers, and the sum of these numbers is 1976.

$\boxed{2}$ We consider the following system with $q = 2p$:

$$\begin{gathered} a_{11}x_1 + \ldots + a_{1q}x_q = 0, \\ a_{21}x_1 + \ldots + a_{2q}x_q = 0, \\ \ldots, \\ a_{p1}x_1 + \ldots + a_{pq}x_q = 0, \end{gathered}$$

in which every coefficient is an element from the set $\{-1, 0, 1\}$. Prove that there exists a solution $x_1, \ldots, x_q$ for the system with the properties:

a.) all $x_j, j = 1, \ldots, q$ are integers;

b.) there exists at least one j for which $x_j \neq 0$;

c.) $|x_j| \leq q$ for any $j = 1, \ldots, q$.

$\boxed{3}$ A sequence (u_n) is defined by

$$u_0 = 2 \quad u_1 = \frac{5}{2}, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \quad \text{for} n = 1, \ldots$$

Prove that for any positive integer n we have

$$[u_n] = 2^{\frac{(2^n - (-1)^n)}{3}}$$

(where [x] denotes the smallest integer $\leq$ x).

IMO 1977

Day 1

1. In the interior of a square $ABCD$ we construct the equilateral triangles ABK, BCL, CDM, DAN. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AK, BK, BL, CL, CM, DM, DN, AN$ are the 12 vertices of a regular dodecagon.

2. In a finite sequence of real numbers the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

3. Let n be a given number greater than 2. We consider the set V_n of all the integers of the form $1 + kn$ with $k = 1, 2, \ldots$ A number m from V_n is called indecomposable in V_n if there are not two numbers p and q from V_n so that $m = pq$. Prove that there exist a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions which differ only in order of the elements of V_n will be considered the same.)

IMO 1977

Day 2

$\boxed{1}$ Let a, b, A, B be given reals. We consider the function defined by

$$f(x) = 1 - a \cdot \cos(x) - b \cdot \sin(x) - A \cdot \cos(2x) - B \cdot \sin(2x).$$

Prove that if for any real number x we have $f(x) \geq 0$ then $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1$.

$\boxed{2}$ Let a, b be two natural numbers. When we divide $a^2 + b^2$ by $a + b$, we the the remainder r and the quotient q. Determine all pairs (a, b) for which $q^2 + r = 1977$.

$\boxed{3}$ Let $\mathbb{N}$ be the set of positive integers. Let f be a function defined on $\mathbb{N}$, which satisfies the inequality $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$. Prove that for any n we have $f(n) = n$.

IMO 1978

Day 1

$\boxed{1}$ Let m and n be positive integers such that $1 \leq m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, so the last three digits of 1978^n. Find m and n such that $m + n$ has its least value.

$\boxed{2}$ We consider a fixed point P in the interior of a fixed sphere. We construct three segments PA, PB, PC, perpendicular two by two, with the vertexes A, B, C on the sphere. We consider the vertex Q which is opposite to P in the parallelepiped (with right angles) with PA, PB, PC as edges. Find the locus of the point Q when A, B, C take all the positions compatible with our problem.

$\boxed{3}$ Let $0 < f(1) < f(2) < f(3) < \dots$ a sequence with all its terms positive. The $n - th$ positive integer which doesn't belong to the sequence is $f(f(n)) + 1$. Find $f(240)$.

IMO 1978

Day 2

$\boxed{1}$ In a triangle ABC we have $AB = AC$. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides AB, AC in the points P, respectively Q. Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC.

$\boxed{2}$ Let f be an injective function from $1, 2, 3, \ldots$ in itself. Prove that for any n we have: $\sum_{k=1}^{n} f(k)k^{-2} \geq \sum_{k=1}^{n} k^{-1}$.

$\boxed{3}$ An international society has its members from six different countries. The list of members contain 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

IMO 1979

Day 1

$\boxed{1}$ If p and q are natural numbers so that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319},$$

prove that p is divisible with 1979.

$\boxed{2}$ We consider a prism which has the upper and inferior basis the pentagons: $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$. Each of the sides of the two pentagons and the segments A_iB_j with $i, j = 1, \ldots, 5$ is colored in red or blue. In every triangle which has all sides colored there exists one red side and one blue side. Prove that all the 10 sides of the two basis are colored in the same color.

$\boxed{3}$ Two circles in a plane intersect. A is one of the points of intersection. Starting simultaneously from A two points move with constant speed, each travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that the two points are always equidistant from P.

IMO 1979

Day 2

$\boxed{1}$ We consider a point P in a plane p and a point $Q \notin p$. Determine all the points R from p for which

$$\frac{QP + PR}{QR}$$

is maximum.

$\boxed{2}$ Determine all real numbers a for which there exists positive reals $x_1, \dots, x_5$ which satisfy the relations $\sum_{k=1}^{5} kx_k = a$, $\sum_{k=1}^{5} k^3 x_k = a^2$, $\sum_{k=1}^{5} k^5 x_k = a^3$.

$\boxed{3}$ Let A and E be opposite vertices of an octagon. A frog starts at vertex A. From any vertex except E it jumps to one of the two adjacent vertices. When it reaches E it stops. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that:

$$a_{2n-1} = 0, \quad a_{2n} = \frac{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}}{\sqrt{2}}.$$

IMO 1980

Austrian-Poland

Day 1

1. Given three infinite arithmetic progressions of natural numbers such that each of the numbers 1,2,3,4,5,6,7 and 8 belongs to at least one of them, prove that the number 1980 also belongs to at least one of them.

2. Let $\{x_n\}$ be a sequence of natural numbers such that
$$(a) 1 = x_1 < x_2 < x_3 < \ldots; \quad (b) x_{2n+1} \le 2n \quad \forall n.$$
Prove that, for every natural number k, there exist terms x_r and x_s such that $x_r - x_s = k$.

3. Prove that the sum of the six angles subtended at an interior point of a tetrahedron by its six edges is greater than 540.

4. Prove that $\sum\{\frac{1}{i_1 i_2 \ldots i_k}\} = n$ is taken over all non-empty subsets $\{i_1, i_2, \ldots, i_k\}$.

5. Let $A_1A_2A_3$ be a triangle and, for $1 \le i \le 3$, let B_i be an interior point of edge opposite A_i. Prove that the perpendicular bisectors of A_iB_i for $1 \le i \le 3$ are not concurrent.

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Day 2

1. Given a sequence $\{a_n\}$ of real numbers such that $|a_{k+m} - a_k - a_m| \le 1$ for all positive integers k and m, prove that, for all positive integers p and q,
$$|\frac{a_p}{p} - \frac{a_q}{q}| < \frac{1}{p} + \frac{1}{q}.$$

2. Find the greatest natural number n such there exist natural numbers $x_1, x_2, \ldots, x_n$ and natural $a_1 < a_2 < \ldots < a_{n-1}$ satisfying the following equations for $i = 1, 2, \ldots, n-1$:
$$x_1 x_2 \ldots x_n = 1980 \quad \text{and} \quad x_i + \frac{1980}{x_i} = a_i.$$

3. Let S be a set of 1980 points in the plane such that the distance between every pair of them is at least 1. Prove that S has a subset of 220 points such that the distance between every pair of them is at least $\sqrt{3}$.

$\boxed{4}$ Let AB be a diameter of a circle; let t_1 and t_2 be the tangents at A and B, respectively; let C be any point other than A on t_1; and let $D_1D_2.E_1E_2$ be arcs on the circle determined by two lines through C. Prove that the lines AD_1 and AD_2 determine a segment on t_2 equal in length to that of the segment on t_2 determined by AE_1 and AE_2.

IMO 1980

Mariehamn (Finland)

Day 1

$\boxed{1}$ Let α, β and γ denote the angles of the triangle ABC. The perpendicular bisector of AB intersects BC at the point X, the perpendicular bisector of AC intersects it at Y. Prove that $\tan(\beta) \cdot \tan(\gamma) = 3$ implies $BC = XY$ (or in other words: Prove that a sufficient condition for $BC = XY$ is $\tan(\beta) \cdot \tan(\gamma) = 3$). Show that this condition is not necessary, and give a necessary and sufficient condition for $BC = XY$.

$\boxed{2}$ Define the numbers $a_0, a_1, \ldots, a_n$ in the following way:

$$a_0 = \frac{1}{2}, \quad a_{k+1} = a_k + \frac{a_k^2}{n} \quad (n > 1, k = 0, 1, \ldots, n-1).$$

Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

$\boxed{3}$ Prove that the equation

$$x^n + 1 = y^{n+1},$$

where n is a positive integer not smaller then 2, has no positive integer solutions in x and y for which x and $n+1$ are relatively prime.

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Day 2

$\boxed{1}$ Determine all positive integers n such that the following statement holds: If a convex polygon with with $2n$ sides $A_1A_2 \ldots A_{2n}$ is inscribed in a circle and $n-1$ of its n pairs of opposite sides are parallel, which means if the pairs of opposite sides

$$(A_1A_2, A_{n+1}A_{n+2}), (A_2A_3, A_{n+2}A_{n+2}), \ldots, (A_{n-1}A_n, A_{2n-1}A_{2n})$$

are parallel, then the sides

$$A_nA_{n+1}, A_{2n}A_1$$

are parallel as well.

$\boxed{2}$ In a rectangular coordinate system we call a horizontal line parallel to the x -axis triangular if it intersects the curve with equation

$$y = x^4 + px^3 + qx^2 + rx + s$$

in the points A, B, C and D (from left to right) such that the segments AB, AC and AD are the sides of a triangle. Prove that the lines parallel to the x - axis intersecting the curve in four distinct points are all triangular or none of them is triangular.

$\boxed{3}$ Find the digits left and right of the decimal point in the decimal form of the number

$$(\sqrt{2} + \sqrt{3})^{1980}.$$

IMO 1980

Mersch (Luxembourg)

Day 1

$\boxed{1}$ The function f is defined on the set $\mathbb{Q}$ of all rational numbers and has values in $\mathbb{Q}$. It satisfies the conditions $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x+y) + 1$ for all $x, y \in \mathbb{Q}$. Determine f.

$\boxed{2}$ Three points A, B, C are such that $B \in]AC[$. On the side of AC we draw the three semicircles with diameters $[AB], [BC]$ and $[AC]$. The common interior tangent at B to the first two semi-circles meets the third circle in E. Let U and V be the points of contact of the common exterior tangent to the first two semi-circles. Denote the are of the triangle ABC as area(ABC). Then please evaluate the ratio $R = area(EUV)/area(EAC)$ as a function of $r_1 = \frac{AB}{2}$ and $r_2 = \frac{BC}{2}$.

$\boxed{3}$ Let p be a prime number. Prove that there is no number divisible by p in the $n-th$ row of Pascal's triangle if and only if n can be represented in the form $n = p^s q - 1$, where s and q are integers with $s \geq 0, 0 < q < p$.

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Day 2

$\boxed{1}$ Two circles C_1 and C_2 are (externally or internally) tangent at a point P. The straight line D is tangent at A to one of the circles and cuts the other circle at the points B and C. Prove that the straight line PA is an interior or exterior bisector of the angle $\angle BPC$.

$\boxed{2}$ Ten gamblers started playing with the same amount of money. Each turn they cast (threw) five dice. At each stage the gambler who had thrown paid to each of his 9 opponents $\frac{1}{n}$ times the amount which that opponent owned at that moment. They threw and paid one after the other. At the 10th round (i.e. when each gambler has cast the five dice once), the dice showed a total of 12, and after payment it turned out that every player had exactly the same sum as he had at the beginning. Is it possible to determine the total shown by the dice at the nine former rounds ?

$\boxed{3}$ Find all pairs of solutions (x, y):

$$x^3 + x^2y + xy^2 + y^3 = 8(x^2 + xy + y^2 + 1).$$

IMO 1981

Day 1

$\boxed{1}$ Consider a variable point P inside a given triangle ABC. Let D, E, F be the feet of the perpendiculars from the point P to the lines BC, CA, AB, respectively. Find all points P which minimize the sum

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}.$$

$\boxed{2}$ Take r such that $1 \le r \le n$, and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each subset has a smallest element. Let $F(n, r)$ be the arithmetic mean of these smallest elements. Prove that:

$$F(n, r) = \frac{n+1}{r+1}.$$

$\boxed{3}$ Determine the maximum value of $m^2 + n^2$, where m and n are integers in the range $1, 2, \ldots, 1981$ satisfying $(n^2 - mn - m^2)^2 = 1$.

IMO 1981

Day 2

$\boxed{1}$ **a.)** For which $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n-1$ numbers?

b.) For which $n > 2$ is there exactly one set having this property?

$\boxed{2}$ Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

$\boxed{3}$ The function $f(x, y)$ satisfies: $f(0, y) = y+1$, $f(x+1, 0) = f(x, 1)$, $f(x+1, y+1) = f(x, f(x+1, y))$ for all non-negative integers x, y. Find $f(4, 1981)$.

IMO 1982

Day 1

1. The function $f(n)$ is defined on the positive integers and takes non-negative integer values. $f(2) = 0, f(3) > 0, f(9999) = 3333$ and for all m, n :

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1.$$

Determine $f(1982)$.

2. A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with the side a_i lying opposite to the vertex A_i. Let M_i be the midpoint of the side a_i, and let T_i be the point where the inscribed circle of triangle $A_1A_2A_3$ touches the side a_i. Denote by S_i the reflection of the point T_i in the interior angle bisector of the angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

3. Consider infinite sequences $\{x_n\}$ of positive reals such that $x_0 = 1$ and $x_0 \geq x_1 \geq x_2 \geq \ldots$.

a.) Prove that for every such sequence there is an $n \geq 1$ such that:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

b.) Find such a sequence such that for all n:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4.$$

IMO 1982

Day 2

$\boxed{1}$ Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

$\boxed{2}$ The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by inner points M and N respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M and N are collinear.

$\boxed{3}$ Let S be a square with sides length 100. Let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ with $A_0 = A_n$. Suppose that for every point P on the boundary of S there is a point of L at a distance from P no greater than $\frac{1}{2}$. Prove that there are two points X and Y of L such that the distance between X and Y is not greater than 1 and the length of the part of L which lies between X and Y is not smaller than 198.

IMO 1983

Day 1

$\boxed{1}$ Find all functions f defined on the set of positive reals which take positive real values and satisfy: $f(xf(y)) = yf(x)$ for all x, y; and $f(x) \to 0$ as $x \to \infty$.

$\boxed{2}$ Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

$\boxed{3}$ Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where x, y, z are non-negative integers.

IMO 1983

Day 2

$\boxed{1}$ Let ABC be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the three segments AB, BC, and CA (including A, B, and C). Determine whether, for every partition of $\mathcal{E}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

$\boxed{2}$ Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?

$\boxed{3}$ Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Determine when equality occurs.

IMO 1984

Day 1

1. Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where x, y and z are non-negative real numbers satisfying $x + y + z = 1$.

2. Find one pair of positive integers a, b such that $ab(a+b)$ is not divisible by 7, but $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

3. Given points O and A in the plane. Every point in the plane is colored with one of a finite number of colors. Given a point X in the plane, the circle $C(X)$ has center O and radius $OX + \frac{\angle AOX}{OX}$, where $\angle AOX$ is measured in radians in the range $[0, 2\pi)$. Prove that we can find a point X, not on OA, such that its color appears on the circumference of the circle $C(X)$.

Day 2

$\boxed{1}$ Let $ABCD$ be a convex quadrilateral with the line CD being tangent to the circle on diameter AB. Prove that the line AB is tangent to the circle on diameter CD if and only if the lines BC and AD are parallel.

$\boxed{2}$ Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices (where $n > 3$). Let p be its perimeter. Prove that:

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right] \cdot \left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x.

$\boxed{3}$ Let a, b, c, d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a+d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.

IMO 1985

Day 1

1 A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

[hide="Historical remark"]*Historical remark.* This problem has been discussed in H. V. Mallison, *Note 1922: Porism of the hexagon*, Mathematical Gazette July 1946, pp. 165-167.

2 Let n and k be relatively prime positive integers with $k < n$. Each number in the set $M = \{1, 2, 3, \ldots, n-1\}$ is colored either blue or white. For each i in M, both i and $n-i$ have the same color. For each $i \neq k$ in M both i and $|i-k|$ have the same color. Prove that all numbers in M must have the same color.

3 For any polynomial $P(x) = a_0 + a_1x + \ldots + a_kx^k$ with integer coefficients, the number of odd coefficients is denoted by $o(P)$. For $i - 0, 1, 2, \ldots$ let $Q_i(x) = (1+x)^i$. Prove that if $i_1, i_2, \ldots, i_n$ are integers satisfying $0 \leq i_1 < i_2 < \ldots < i_n$, then:

$$o(Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n}) \geq o(Q_{i_1}).$$

IMO 1985

Day 2

1. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 23, prove that M contains a subset of 4 elements whose product is the 4th power of an integer.

2. A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90^\circ$.

3. For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting:

$$x_{n+1} = x_n(x_n + \frac{1}{n}).$$

Prove that there exists exactly one value of x_1 which gives $0 < x_n < x_{n+1} < 1$ for all n.

IMO 1986

Day 1

1 Let d be any positive integer not equal to $2, 5$ or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

2 Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \geq 4$. Construct a set of points $P_1, P_2, P_3, \ldots$ such that P_{k+1} is the image of P_k under a rotation center A_{k+1} through an angle 120^o clockwise for $k = 0, 1, 2, \ldots$. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

3 To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

IMO 1986

Day 2

$\boxed{1}$ Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) with center O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, with X remaining inside the polygon. Find the locus of X.

$\boxed{2}$ Find all functions f defined on the non-negative reals and taking non-negative real values such that: $f(2) = 0, f(x) \neq 0$ for $0 \leq x < 2$, and $f(xf(y))f(y) = f(x + y)$ for all x, y.

$\boxed{3}$ Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line L parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on L is not greater than 1?

IMO 1987

Day 1

$\boxed{1}$ Let $p_n(k)$ be the number of permutations of the set $\{1, 2, 3, \ldots, n\}$ which have exactly k fixed points. Prove that $\sum_{k=0}^{n} kp_n(k) = n!$.

$\boxed{2}$ In an acute-angled triangle ABC the interior bisector of angle A meets BC at L and meets the circumcircle of ABC again at N. From L perpendiculars are drawn to AB and AC, with feet K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.

$\boxed{3}$ Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i| \leq k - 1$ for all i, and $|a_1x_1 + a_2x_2 + \ldots + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}$.

IMO 1987

Day 2

$\boxed{1}$ Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for all n.

$\boxed{2}$ Let $n \geq 3$ be an integer. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

$\boxed{3}$ Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

IMO 1988

Day 1

$\boxed{1}$ Consider 2 concentric circle radii R and r ($R > r$) with centre O. Fix P on the small circle and consider the variable chord PA of the small circle. Points B and C lie on the large circle; B, P, C are collinear and BC is perpendicular to AP.

i.) For which values of $\angle OPA$ is the sum $BC^2 + CA^2 + AB^2$ extremal?

ii.) What are the possible positions of the midpoints U of BA and V of AC as $\angle OPA$ varies?

$\boxed{2}$ Let n be an even positive integer. Let $A_1, A_2, \ldots, A_{n+1}$ be sets having n elements each such that any two of them have exactly one element in common while every element of their union belongs to at least two of the given sets. For which n can one assign to every element of the union one of the numbers 0 and 1 in such a manner that each of the sets has exactly $\frac{n}{2}$ zeros?

$\boxed{3}$ A function f defined on the positive integers (and taking positive integers values) is given by:

$$\begin{aligned} f(1) = 1, f(3) &= 3 \\ f(2 \cdot n) &= f(n) \\ f(4 \cdot n + 1) &= 2 \cdot f(2 \cdot n + 1) - f(n) \\ f(4 \cdot n + 3) &= 3 \cdot f(2 \cdot n + 1) - 2 \cdot f(n), \end{aligned}$$

for all positive integers n. Determine with proof the number of positive integers ≤ 1988 for which $f(n) = n$.

IMO 1988

Day 2

1. Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose length is 1988.

2. In a right-angled triangle ABC let AD be the altitude drawn to the hypotenuse and let the straight line joining the incentres of the triangles ABD, ACD intersect the sides AB, AC at the points K, L respectively. If E and E_1 dnote the areas of triangles ABC and AKL respectively, show that

$$\frac{E}{E_1} \geq 2.$$

3. Let a and b be two positive integers such that $a \cdot b + 1$ divides $a^2 + b^2$. Show that $\frac{a^2+b^2}{a \cdot b+1}$ is a perfect square.

IMO 1989

Day 1

$\boxed{1}$ Prove that in the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets $A_i, \{i = 1, 2, \ldots, 117\}$ such that

i.) each A_i contains 17 elements

ii.) the sum of all the elements in each A_i is the same.

$\boxed{2}$ ABC is a triangle, the bisector of angle A meets the circumcircle of triangle ABC in A_1, points B_1 and C_1 are defined similarly. Let AA_1 meet the lines that bisect the two external angles at B and C in A_0. Define B_0 and C_0 similarly. Prove that the area of triangle $A_0B_0C_0 = 2\cdot$ area of hexagon $AC_1BA_1CB_1 \geq 4\cdot$ area of triangle ABC.

$\boxed{3}$ Let n and k be positive integers and let S be a set of n points in the plane such that

i.) no three points of S are collinear, and

ii.) for every point P of S there are at least k points of S equidistant from P.

Prove that:

$$k < \frac{1}{2} + \sqrt{2 \cdot n}$$

IMO 1989

Day 2

$\boxed{4}$ Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$

$\boxed{5}$ Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

$\boxed{6}$ A permutation $\{x_1, \ldots, x_{2n}\}$ of the set $\{1, 2, \ldots, 2n\}$ where n is a positive integer, is said to have property T if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that, for each n, there are more permutations with property T than without.

IMO 1990

Day 1

$\boxed{1}$ Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines BC and AC at F and G, respectively. If

$$\frac{AM}{AB} = t,$$

find $\frac{EG}{EF}$ in terms of t.

$\boxed{2}$ Let $n \geq 3$ and consider a set E of $2n - 1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is **good** if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

$\boxed{3}$ Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

IMO 1990

Day 2

$\boxed{1}$ Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y in $\mathbb{Q}^+$.

$\boxed{2}$ Given an initial integer $n_0 > 1$, two players, $\mathcal{A}$ and $\mathcal{B}$, choose integers n_1, n_2, n_3, $\ldots$ alternately according to the following rules :

I.) Knowing n_{2k}, $\mathcal{A}$ chooses any integer n_{2k+1} such that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$

II.) Knowing n_{2k+1}, $\mathcal{B}$ chooses any integer n_{2k+2} such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power.

Player $\mathcal{A}$ wins the game by choosing the number 1990; player $\mathcal{B}$ wins by choosing the number 1. For which n_0 does :

a.) $\mathcal{A}$ have a winning strategy? **b.)** $\mathcal{B}$ have a winning strategy? **c.)** Neither player have a winning strategy?

$\boxed{3}$ Prove that there exists a convex 1990-gon with the following two properties :

a.) All angles are equal. **b.)** The lengths of the 1990 sides are the numbers 1^2, 2^2, 3^2, $\cdots$, 1990^2 in some order.

IMO 1991

Day 1

$\boxed{1}$ Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

$\boxed{2}$ Let $n > 6$ be an integer and $a_1, a_2, \cdots, a_k$ be all the natural numbers less than n and relatively prime to n. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

$\boxed{3}$ Let $S = \{1, 2, 3, \cdots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

IMO 1991

Day 2

$\boxed{1}$ Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

[hide="Graph-Definition"]A **graph** consists of a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of vertices u, v belongs to at most one edge. The graph G is connected if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, \cdots, v_m = y$ such that each pair v_i, v_{i+1} $(0 \leq i < m)$ is joined by an edge of G.

$\boxed{2}$ Let ABC be a triangle and P an interior point of ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to 30°.

$\boxed{3}$ An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be **bounded** if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j||i - j|^a \geq 1$$

for every pair of distinct nonnegative integers i, j.

IMO 1992

Day 1

1. Find all integers a, b, c with $1 < a < b < c$ such that
$$(a-1)(b-1)(c-1)$$
is a divisor of $abc - 1$.

2. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f\left(x^2 + f(y)\right) = y + (f(x))^2 \qquad \text{for all}\, x, y \in \mathbb{R}.$$

3. Consider 9 points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

IMO 1992

Day 2

$\boxed{1}$ In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

$\boxed{2}$ Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set A.

[hide="Note"] Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.

$\boxed{3}$ For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.

a.) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$. **b.)** Find an integer n such that $S(n) = n^2 - 14$. **c.)** Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.

IMO 1993

Day 1

$\boxed{1}$ Let $n > 1$ be an integer and let $f(x) = x^n + 5 \cdot x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that $f(x) = g(x) \cdot h(x)$.

$\boxed{2}$ Let A, B, C, D be four points in the plane, with C and D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90^\circ + \angle ACB$. Find the ratio

$$\frac{AB \cdot CD}{AC \cdot BD},$$

and prove that the circumcircles of the triangles ACD and BCD are orthogonal. (Intersecting circles are said to be orthogonal if at either common point their tangents are perpendicuar. Thus, proving that the circumcircles of the triangles ACD and BCD are orthogonal is equivalent to proving that the tangents to the circumcircles of the triangles ACD and BCD at the point C are perpendicular.)

$\boxed{3}$ On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

Day 2

$\boxed{4}$ For three points A, B, C in the plane, we define $m(ABC)$ to be the smallest length of the three heights of the triangle ABC, where in the case A, B, C are collinear, we set $m(ABC) = 0$. Let A, B, C be given points in the plane. Prove that for any point X in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

$\boxed{5}$ Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine if there exists a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$ with the following properties:

(i) $f(1) = 2$;

(ii) $f(f(n)) = f(n) + n, (n \in \mathbb{N})$.

$\boxed{6}$ Let $n > 1$ be an integer. In a circular arrangement of n lamps $L_0, \ldots, L_{n-1}$, each of of which can either ON or OFF, we start with the situation where all lamps are ON, and then carry out a sequence of steps, $Step_0, Step_1, \ldots$. If L_{j-1} (j is taken mod n) is ON then $Step_j$ changes the state of L_j (it goes from ON to OFF or from OFF to ON) but does not change the state of any of the other lamps. If L_{j-1} is OFF then $Step_j$ does not change anything at all. Show that:

(i) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again,

(ii) If n has the form 2^k then all the lamps are ON after $n^2 - 1$ steps,

(iii) If n has the form $2^k + 1$ then all lamps are ON after $n^2 - n + 1$ steps.

IMO 1994

Hong-Kong

Day 1 - *13 July 1994*

$\boxed{1}$ Let m and n be two positive integers. Let a_1, a_2, $\ldots$, a_m be m different numbers from the set $\{1, 2, \ldots, n\}$ such that for any two indices i and j with $1 \leq i \leq j \leq m$ and $a_i + a_j \leq n$, there exists an index k such that $a_i + a_j = a_k$. Show that

$$\frac{a_1 + a_2 + ... + a_m}{m} \geq \frac{n+1}{2}.$$

$\boxed{2}$ Let ABC be an isosceles triangle with $AB = AC$. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB. Q is an arbitrary point on BC different from B and C. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear. Prove that OQ is perpendicular to EF if and only if $QE = QF$.

$\boxed{3}$ For any positive integer k, let f_k be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation contains exactly three 1s.

(a) Prove that for any positive integer m, there exists at least one positive integer k such that $f(k) = m$.

(b) Determine all positive integers m for which there exists *exactly one* k with $f(k) = m$.

IMO 1994

Hong-Kong

Day 2 - *14 July 1994*

$\boxed{4}$ Find all ordered pairs (m, n) where m and n are positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.

$\boxed{5}$ Let S be the set of all real numbers strictly greater than 1. Find all functions $f : S \to S$ satisfying the two conditions:

(a) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x, y in S;

(b) $\frac{f(x)}{x}$ is strictly increasing on each of the two intervals $-1 < x < 0$ and $0 < x$.

$\boxed{6}$ Show that there exists a set A of positive integers with the following property: for any infinite set S of primes, there exist *two* positive integers m in A and n not in A, each of which is a product of k distinct elements of S for some $k \geq 2$.

IMO 1995

York University, North York, Ontario

Day 1 - *19 July 1995*

$\boxed{1}$ Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

$\boxed{2}$ Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3\,(b+c)} + \frac{1}{b^3\,(c+a)} + \frac{1}{c^3\,(a+b)} \geq \frac{3}{2}.$$

$\boxed{3}$ Determine all integers $n > 3$ for which there exist n points $A_1, \cdots, A_n$ in the plane, no three collinear, and real numbers $r_1, \cdots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_i A_j A_k$ is $r_i + r_j + r_k$.

IMO 1995

York University, North York, Ontario

Day 2 - *20 July 1995*

4 Find the maximum value of x_0 for which there exists a sequence $x_0, x_1 \cdots, x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i},$$

for all $i = 1, \cdots, 1995$.

5 Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$.

6 Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \cdots\ 2p\}$ are there, the sum of whose elements is divisible by p?

IMO 1996

Mumbai, India

Day 1 - *10 July 1996*

1 We are given a positive integer r and a rectangular board $ABCD$ with dimensions $AB = 20, BC = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.

(a) Show that the task cannot be done if r is divisible by 2 or 3.

(b) Prove that the task is possible when $r = 73$.

(c) Can the task be done when $r = 97$?

2 Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.

3 Let $\mathbb{N}_0$ denote the set of nonnegative integers. Find all functions f from $\mathbb{N}_0$ to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \qquad \text{for all } m, n \in \mathbb{N}_0.$$

IMO 1996

Mumbai, India

Day 2 - *11 July 1996*

$\boxed{4}$ The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

$\boxed{5}$ Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

$\boxed{6}$ Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \cdots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions :

(a) $x_0 = x_n = 0$, and

(b) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

IMO 1997

Mar del Plata, Argentina

Day 1 - *24 July 1997*

$\boxed{1}$ In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

c) Show that there is no constant $C \in \mathbb{R}$ such that $f(m,n) < C$ for all m and n.

$\boxed{2}$ It is known that $\angle BAC$ is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T.

Show that $AU = TB + TC$.

Alternative formulation:

Four different points A, B, C, D are chosen on a circle Γ such that the triangle BCD is not right-angled. Prove that:

(a) The perpendicular bisectors of AB and AC meet the line AD at certain points W and V, respectively, and that the lines CV and BW meet at a certain point T.

(b) The length of one of the line segments AD, BT, and CT is the sum of the lengths of the other two.

$\boxed{3}$ Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions:

$$\begin{cases} |x_1 + x_2 + \cdots + x_n| & = & 1 \\ |x_i| & \le & \dfrac{n+1}{2} \qquad \text{for } i = 1, 2, \ldots, n. \end{cases}$$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

IMO 1997

Mar del Plata, Argentina

Day 2 - *25 July 1997*

$\boxed{4}$ An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver matrix* if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

$\boxed{5}$ Find all pairs (a, b) of positive integers that satisfy the equation: $a^{b^2} = b^a$.

$\boxed{6}$ For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways: 4; 2+2; 2+1+1; 1+1+1+1.

Prove that, for any integer $n \geq 3$ we have $2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$.

IMO 1998

Taipei, Taiwan

Day 1 - *15 June 1998*

$\boxed{1}$ A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

$\boxed{2}$ In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

$\boxed{3}$ For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that $\frac{\tau(n^2)}{\tau(n)} = m$.

IMO 1998

Taipei, Taiwan

Day 2 - *16 June 1998*

$\boxed{4}$ Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

$\boxed{5}$ Let I be the incenter of triangle ABC. Let K, L and M be the points of tangency of the incircle of ABC with AB, BC and CA, respectively. The line t passes through B and is parallel to KL. The lines MK and ML intersect t at the points R and S. Prove that $\angle RIS$ is acute.

$\boxed{6}$ Determine the least possible value of $f(1998)$, where f is a function from the set $\mathbf{N}$ of positive integers into itself such that for all $m, n \in \mathbf{N}$,

$$f\left(n^2 f(m)\right) = m\left[f(n)\right]^2.$$

IMO 1999

Bucharest, Romania

Day 1

$\boxed{1}$ A set S of points from the space will be called **completely symmetric** if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

$\boxed{2}$ Let $n \geq 2$ be a fixed integer. Find the least constant C such the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_i x_i\right)^4$$

holds for any $x_1, \ldots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality.

$\boxed{3}$ Let b be an even positive integer. We say that two different cells of a $n \times n$ board are **neighboring** if they have a common side. Find the minimal number of cells on he $n \times n$ board that must be marked so that any cell marked or not marked) has a marked neighboring cell.

IMO 1999

Bucharest, Romania

Day 2

$\boxed{4}$ Find all the pairs of positive integers (x, p) such that p is a prime, $x \leq 2p$ and x^{p-1} is a divisor of $(p-1)^x + 1$.

$\boxed{5}$ Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

$\boxed{6}$ Find all the functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

IMO 2000

Seul, Korea

Day 1

1 Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

2 Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

3 Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$.

Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

Day 2

$\boxed{4}$ A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

How many ways are there to put the cards in the three boxes so that the trick works?

$\boxed{5}$ Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?

$\boxed{6}$ Let AH_1, BH_2, CH_3 be the altitudes of an acute angled triangle ABC. Its incircle touches the sides BC, AC and AB at T_1, T_2 and T_3 respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3 and H_3H_1 with respect to the lines T_1T_2, T_2T_3 and T_3T_1. Prove that these images form a triangle whose vertices lie on the incircle of ABC.

IMO 2001

Washington DC, USA

Day 1 - *06 July 2001*

$\boxed{1}$ Consider an acute-angled triangle ABC. Let P be the foot of the altitude of triangle ABC issuing from the vertex A, and let O be the circumcenter of triangle ABC. Assume that $\angle C \geq \angle B + 30^\circ$. Prove that $\angle A + \angle COP < 90^\circ$.

$\boxed{2}$ Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

$\boxed{3}$ Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

IMO 2001

Washington DC, USA

Day 2 - *07 July 2001*

$\boxed{4}$ Let n be an odd integer greater than 1 and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist permutations $a \neq b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

$\boxed{5}$ Let ABC be a triangle with $\angle BAC = 60^\circ$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

$\boxed{6}$ Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

IMO 2002

Glasgow, United Kingdom

Day 1 - *24 July 2002*

$\boxed{1}$ Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

$\boxed{2}$ The circle S has centre O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.

$\boxed{3}$ Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

Laurentiu Panaitopol, Romania

Day 2 - *25 July 2002*

$\boxed{4}$ Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \ldots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.

$\boxed{5}$ Find all functions f from the reals to the reals such that

$$(f(x) + f(z))\,(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real x, y, z, t.

$\boxed{6}$ Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centres $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

IMO 2003

Tokio, Japan

Day 1 - *13 July 2003*

$\boxed{1}$ Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers t_1, $t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

$\boxed{2}$ Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

$\boxed{3}$ Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\dfrac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

IMO 2003

Tokio, Japan

Day 2 - *14 July 2003*

$\boxed{4}$ Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

$\boxed{5}$ Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

$\boxed{6}$ Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

IMO 2004

Athens, Greece

Day 1 - *12 July 2004*

1 1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

2 Find all polynomials f with real coefficients such that for all reals a, b, c such that $ab+bc+ca = 0$ we have the following relations

$$f(a-b) + f(b-c) + f(c-a) = 2f(a+b+c).$$

3 Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure. [img]http://www.mathlinks.ro/Forum/files/hook.png[/img] Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that

- the rectangle is covered without gaps and without overlaps - no part of a hook covers area outside the rectagle.

IMO 2004

Athens, Greece

Day 2 - *13 July 2004*

$\boxed{4}$ Let $n \geq 3$ be an integer. Let t_1, t_2, ..., t_n be positive real numbers such that

$n^2 + 1 > (t_1 + t_2 + ... + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + ... + \frac{1}{t_n}\right)$.

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

$\boxed{5}$ In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$.

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

$\boxed{6}$ We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers n such that n has a multiple which is alternating.

IMO 2005

Merida, Yutacan, Mexico

Day 1 - *13 July 2005*

$\boxed{1}$ Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Bogdan Enescu, Romania

$\boxed{2}$ Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n.

Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

$\boxed{3}$ Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Hojoo Lee, Korea

Day 2 - *14 July 2005*

$\boxed{4}$ Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

$\boxed{5}$ Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.

Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

$\boxed{6}$ In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

Radu Gologan and Dan Schwartz

IMO 2006

Slovenia

Day 1 - *12 July 2006*

$\boxed{1}$ Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

$\boxed{2}$ Let P be a regular 2006-gon. A diagonal is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

$\boxed{3}$ Determine the least real number M such that the inequality $\left|ab\left(a^2-b^2\right)+bc\left(b^2-c^2\right)+ca\left(c^2-a^2\right)\right| \leq M \,($ holds for all real numbers a, b and c.

IMO 2006

Slovenia

Day 2 - *13 July 2006*

$\boxed{4}$ Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

$\boxed{5}$ Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

$\boxed{6}$ Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

IMO 2007

Ha Noi, Vietnam

Day 1 - *25 July 2007*

1. Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i, $(1 \le i \le n)$, define

$$d_i = \max\{a_j \mid 1 \le j \le i\} - \min\{a_j \mid i \le j \le n\}$$

and let $d = \max\{d_i \mid 1 \le i \le n\}$.

(a) Prove that, for any real numbers $x_1 \le x_2 \le \cdots \le x_n$,

$$\max\{|x_i - a_i| \mid 1 \le i \le n\} \ge \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \le x_2 \le \cdots \le x_n$ such that the equality holds in (*).

Author: Michael Albert, New Zealand

2. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

Author: Charles Leytem, Luxembourg

3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Author: Vasily Astakhov, Russia

Day 2 - *26 July 2007*

$\boxed{4}$ In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Author: Marek Pechal, Czech Republic

$\boxed{5}$ Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

[hide="How a lemma of an ISL problem was selected for IMO"]Strictly this IMO problem does not correspond to any ISL problem 2007. This is rather a lemma of ISL 2007, number theory problem N6. But as the IMO Problem Selection Committee appreciated this problem so much they chose to select this lemma as IMO problem, re-classifying [url=http://www.mathlinks.ro/viewtopic.php? Shortlist Number Theory Problem N6[/url] by just using its key lemma from hard to medium. [url=http://www.imo-register.org.uk/2007-report.html]Source: UK IMO Report.[/url]

Edited by Orlando Dhring

Author: Kevin Buzzard and Edward Crane, United Kingdom

$\boxed{6}$ Let n be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

Author: Gerhard Wginger, Netherlands

IMO 2008

Madrid, Spain

Day 1 - *16 July 2008*

$\boxed{1}$ Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

Prove that six points A_1 , A_2, B_1, B_2, C_1 and C_2 are concyclic.

Author: Andrey Gavrilyuk, Russia

$\boxed{2}$ **(i)** If x, y and z are three real numbers, all different from 1, such that $xyz = 1$, then prove that $\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$. (With the $\sum$ sign for cyclic summation, this inequality could be rewritten as $\sum \frac{x^2}{(x-1)^2} \geq 1$.)

(ii) Prove that equality is achieved for infinitely many triples of rational numbers x, y and z.

Author: Walther Janous, Austria

$\boxed{3}$ Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

Author: Kestutis Cesnavicius, Lithuania

IMO 2008

Madrid, Spain

Day 2 - *17 July 2008*

$\boxed{1}$ Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so f is a function from the positive real numbers) such that

$$\frac{\left(f(w)\right)^2 + \left(f(x)\right)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbes w, x, y, z, satisfying $wx = yz$.

Author: Hojoo Lee, South Korea

$\boxed{2}$ Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off.

Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine $\frac{N}{M}$.

Author: Bruno Le Floch and Ilia Smilga, France

$\boxed{3}$ Let $ABCD$ be a convex quadrilateral with BA different from BC. Denote the incircles of triangles ABC and ADC by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD.

Prove that the common external tangents to k_1 and k_2 intersects on k.

Author: Vladimir Shmarov, Russia

IMO 2009

Bremen, Germany

Day 1 - *15 July 2009*

$\boxed{1}$ Let n be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $1, 2, \ldots, n$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

Proposed by Ross Atkins, Australia

$\boxed{2}$ Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Proposed by Sergei Berlov, Russia

$\boxed{3}$ Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences $s_{s_1}, s_{s_2}, s_{s_3}, \ldots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$ are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Proposed by Gabriel Carroll, USA

IMO 2009

Bremen, Germany

Day 2

$\boxed{1}$ Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E , respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

Jan Vonk, Belgium, Peter Vandendriessche, Belgium and Hojoo Lee, Korea

$\boxed{2}$ Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

Proposed by Bruno Le Floch, France

$\boxed{3}$ Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

Proposed by Dmitry Khramtsov, Russia

IMO 2010

Day 1

$\boxed{1}$ Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

Proposed by Pierre Bornsztein, France

$\boxed{2}$ Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Proposed by Tai Wai Ming and Wang Chongli, Hong Kong

$\boxed{3}$ Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is perfect square for all $m, n \in \mathbb{N}$.

Proposed by Gabriel Carroll, USA

IMO 2010

Day 2

1 Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

2 Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

Type 1) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

Type 2) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

3 Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and s be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \text{ for all } n > s.$$

Prove there exist positive integers $\ell \leq s$ and N, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \geq N.$$

Proposed by Morteza Saghafiyan, Iran

IMO 2011

Day 1

$\boxed{1}$ Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

Proposed by Fernando Campos, from Mexico

$\boxed{2}$ Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Proposed by Geoffrey Smith, United Kingdom

$\boxed{3}$ Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x + y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Proposed by Igor Voronovich, Belarus

IMO 2011

Day 2

1. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

   Proposed by Morteza Saghafian, Iran

2. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

   Proposed by Mahyar Sefidgaran, Iran

3. Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

   Proposed by Japan